Exercise Set 4.1*

In 1–3, use the definitions of even, odd, prime, and composite to justify each of your answers.

1. Assume that k is a particular integer.
a. Is -17 an odd integer? b. Is 0 an even integer?
c. Is $2k - 1$ odd?

2. Assume that m and n are particular integers.
a. Is $6m + 8n$ even? b. Is $10mn + 7$ odd?
c. If $m > n > 0$, is $m^2 - n^2$ composite?

3. Assume that r and s are particular integers.
a. Is $4rs$ even? b. Is $6r + 4s^2 + 3$ odd?
c. If r and s are both positive, is $r^2 + 2rs + s^2$ composite?

Prove the statements in 4–10.

4. There are integers m and n such that $m > 1$ and $n > 1$ and $\frac{1}{m} + \frac{1}{n}$ is an integer.

5. There are distinct integers m and n such that $\frac{1}{m} + \frac{1}{n}$ is an integer.

6. There are real numbers a and b such that

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}.$$

7. There is an integer $n > 5$ such that $2^n - 1$ is prime.

8. There is a real number x such that $x > 1$ and $2^x > x^{10}$.

> **Definition:** An integer n is called a **perfect square** if, and only if, $n = k^2$ for some integer k.

9. There is a perfect square that can be written as a sum of two other perfect squares.

10. There is an integer n such that $2n^2 - 5n + 2$ is prime.

Disprove the statements in 11–13 by giving a counterexample.

11. For all real numbers a and b, if $a < b$ then $a^2 < b^2$.

12. For all integers n, if n is odd then $\frac{n-1}{2}$ is odd.

13. For all integers m and n, if $2m + n$ is odd then m and n are both odd.

In 14–16, determine whether the property is true for all integers, true for no integers, or true for some integers and false for other integers. Justify your answers.

14. $(a+b)^2 = a^2 + b^2$ ***H*** 15. $-a^n = (-a)^n$

16. The average of any two odd integers is odd.

Prove the statements in 17 and 18 by the method of exhaustion.

17. Every positive even integer less than 26 can be expressed as a sum of three or fewer perfect squares. (For instance, $10 = 1^2 + 3^2$ and $16 = 4^2$.)

18. For each integer n with $1 \leq n \leq 10$, $n^2 - n + 11$ is a prime number.

19. a. Rewrite the following theorem in three different ways: as ∀ _____, if _____ then _____, as ∀ _____, _____ (without using the words *if* or *then*), and as If _____, then _____ (without using an explicit universal quantifier).
b. Fill in the blanks in the proof of the theorem.

Theorem: The sum of any even integer and any odd integer is odd.

Proof: Suppose m is any even integer and n is $\underline{(a)}$. By definition of even, $m = 2r$ for some $\underline{(b)}$, and by definition of odd, $n = 2s + 1$ for some integer s. By substitution and algebra,

$$m + n = \underline{(c)} = 2(r + s) + 1.$$

Since r and s are both integers, so is their sum $r + s$. Hence $m + n$ has the form twice some integer plus one, and so $\underline{(d)}$ by definition of odd.

Each of the statements in 20–23 is true. For each, (a) rewrite the statement with the quantification implicit as If _____, then _____, and (b) write the first sentence of a proof (the "starting point") and the last sentence of a proof (the "conclusion to be shown"). Note that you do not need to understand the statements in order to be able to do these exercises.

20. For all integers m, if $m > 1$ then $0 < \frac{1}{m} < 1$.

21. For all real numbers x, if $x > 1$ then $x^2 > x$.

22. For all integers m and n, if $mn = 1$ then $m = n = 1$ or $m = n = -1$.

23. For all real numbers x, if $0 < x < 1$ then $x^2 < x$.

*For exercises with blue numbers, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or partial solution is given. The symbol ✶ signals that an exercise is more challenging than usual.

Prove the statements in 24–34. In each case use only the definitions of the terms and the Assumptions listed on page 146, not any previously established properties of odd and even integers. Follow the directions given in this section for writing proofs of universal statements.

24. The negative of any even integer is even.

25. The difference of any even integer minus any odd integer is odd.

H **26.** The difference between any odd integer and any even integer is odd. (*Note:* The "proof" shown in exericse 39 contains an error. Can you spot it?)

27. The sum of any two odd integers is even.

28. For all integers n, if n is odd then n^2 is odd.

29. For all integers n, if n is odd then $3n+5$ is even.

30. For all integers m, if m is even then $3m+5$ is odd.

31. If k is any odd integer and m is any even integer, then, k^2+m^2 is odd.

32. If a is any odd integer and b is any even integer, then, $2a+3b$ is even.

33. If n is any even integer, then $(-1)^n = 1$.

34. If n is any odd integer, then $(-1)^n = -1$.

Prove that the statements in 35–37 are false.

35. There exists an integer $m \geq 3$ such that $m^2 - 1$ is prime.

36. There exists an integer n such that $6n^2 + 27$ is prime.

37. There exists an integer $k \geq 4$ such that $2k^2 - 5k + 2$ is prime.

Find the mistakes in the "proofs" shown in 38–42.

38. Theorem: For all integers k, if $k > 0$ then k^2+2k+1 is composite.

"**Proof:** For $k = 2$, $k^2+2k+1 = 2^2+2\cdot 2+1 = 9$. But $9 = 3\cdot 3$, and so 9 is composite. Hence the theorem is true."

39. Theorem: The difference between any odd integer and any even integer is odd.

"**Proof:** Suppose n is any odd integer, and m is any even integer. By definition of odd, $n = 2k+1$ where k is an integer, and by definition of even, $m = 2k$ where k is an integer. Then

$$n - m = (2k+1) - 2k = 1.$$

But 1 is odd. Therefore, the difference between any odd integer and any even integer is odd."

40. Theorem: For all integers k, if $k > 0$ then k^2+2k+1 is composite.

"**Proof:** Suppose k is any integer such that $k > 0$. If k^2+2k+1 is composite, then $k^2+2k+1 = rs$ for some integers r and s such that

$$1 < r < (k^2+2k+1)$$

and $$1 < s < (k^2+2k+1).$$

Since $$k^2+2k+1 = rs$$

and both r and s are strictly between 1 and k^2+2k+1, then k^2+2k+1 is not prime. Hence k^2+2k+1 is composite as was to be shown."

41. **Theorem:** The product of an even integer and an odd integer is even.

"**Proof:** Suppose m is an even integer and n is an odd integer. If $m\cdot n$ is even, then by definition of even there exists an integer r such that $m\cdot n = 2r$. Also since m is even, there exists an integer p such that $m = 2p$, and since n is odd there exists an integer q such that $n = 2q+1$. Thus

$$mn = (2p)(2q+1) = 2r,$$

where r is an integer. By definition of even, then, $m\cdot n$ is even, as was to be shown."

42. **Theorem:** The sum of any two even integers equals $4k$ for some integer k.

"**Proof:** Suppose m and n are any two even integers. By definition of even, $m = 2k$ for some integer k and $n = 2k$ for some integer k. By substitution,

$$m + n = 2k + 2k = 4k.$$

This is what was to be shown."

In 43–60 determine whether the statement is true or false. Justify your answer with a proof or a counterexample, as appropriate. In each case use only the definitions of the terms and the Assumptions listed on page 146 not any previously established properties.

43. The product of any two odd integers is odd.

44. The negative of any odd integer is odd.

45. The difference of any two odd integers is odd.

46. The product of any even integer and any integer is even.

47. If a sum of two integers is even, then one of the summands is even. (In the expression $a+b$, a and b are called **summands.**)

48. The difference of any two even integers is even.

49. The difference of any two odd integers is even.

50. For all integers n and m, if $n - m$ is even then $n^3 - m^3$ is even.

51. For all integers n, if n is prime then $(-1)^n = -1$.

52. For all integers m, if $m > 2$ then $m^2 - 4$ is composite.

53. For all integers n, $n^2 - n + 11$ is a prime number.

54. For all integers n, $4(n^2+n+1) - 3n^2$ is a perfect square.

55. Every positive integer can be expressed as a sum of three or fewer perfect squares.

H ✶ **56.** (Two integers are **consecutive** if, and only if, one is one more than the other.) Any product of four consecutive integers is one less than a perfect square.

57. If m and n are positive integers and mn is a perfect square, then m and n are perfect squares.

58. The difference of the squares of any two consecutive integers is odd.

59. For all nonnegative real numbers a and b, $\sqrt{ab} = \sqrt{a}\sqrt{b}$. (Note that if x is a nonnegative real number, then there is a unique nonnegative real number y, denoted $\sqrt{x}$, such that $y^2 = x$.)

60. For all nonnegative real numbers a and b,

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}.$$

61. Suppose that integers m and n are perfect squares. Then $m + n + 2\sqrt{mn}$ is also a perfect square. Why?

H ✶ **62.** If p is a prime number, must $2^p - 1$ also be prime? Prove or give a counterexample.

✶ 63. If n is a nonnegative integer, must $2^{2^n} + 1$ be prime? Prove or give a counterexample.