M252 Formula Recitation
Fall 2007 chapters 13 - 15

For $z = f(x,y)$ the total differential is

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \quad \text{or} \quad \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

For $w = f(x,y)$, $x = g(t)$, $y = h(t)$

the chain rule becomes

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

For $w = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$

the chain rule becomes

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

For $z = f(x,y)$ the gradient of f is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \quad \text{or} \quad \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

The directional derivative of $f(x,y)$ in the direction of the unit vector $\vec{u}$ is

$$D_u f(x,y) = \nabla f \cdot \vec{u}$$

A normal vector to the surface $f(x,y,z)=0$ at the point (x_0, y_0, z_0) is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

The equation of the tangent plane to the surface $f(x,y,z)=0$ at the point (x_0, y_0, z_0) is

$$\nabla f \cdot (\langle x,y,z\rangle - \langle x_0, y_0, z_0\rangle) = 0$$

$$\frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) + \frac{\partial f}{\partial z}(z-z_0) = 0$$

The area of the surface $z = f(x,y)$ over the region R is

$$\text{surface area} = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$

Definition of Conservative Vector Field.
A vector field F is called conservative if there is a scalar valued function f for which $\nabla f = \vec{F}$.

Test for Conservative vector field.

$\vec{F}(x,y) = \langle M, N \rangle$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

The curl of $\vec{F}(x,y,z) = \langle M, N, P \rangle$ is defined as

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

Test for Conservative Vector Field.

$\vec{F}(x,y,z) = \langle M, N, P \rangle$ is conservative

if and only if

curl $\vec{F} = \vec{0}$ i.e, $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

The divergence of the vector field

$\vec{F}(x,y,z) = \langle M, N, P \rangle$ is

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

For the smooth curve C given by

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ $\quad a \le t \le b$

the line integral becomes

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

For $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

the arc length differential becomes

$$ds = \|\vec{r}\,'(t)\|\,dt \quad \text{ie.} \quad \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

For the smooth curve C given by $\vec{r}(t)$, $a \le t \le b$ the line integral of the vectorfield $\vec{F}$ in terms of $\vec{r}$, s, and t is

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F} \cdot \vec{r}'(t)\, dt$$

The Fundamental Theorem of Line Integrals for the conservative vectorfield $F(x,y) = M\vec{i} + N\vec{j}$ and the piecewise smooth curve C given by $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$ $a \le t \le b$ states

$$\int_C \vec{F} \cdot d\vec{r} = f(x(b), y(b)) - f(x(a), y(a)) \quad \text{where } \nabla f = \vec{F}$$

Let R be a simply connected region with a piecewise smooth boundary C, oriented clockwise, then Green's Theorem states

$$\int_C M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA$$