

M252

# Formula Recitation

chapters 12-14

For  $z = f(x, y)$  the total differential is

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad \text{or} \quad \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$


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For  $w = f(x, y)$ ,  $x = g(t)$ ,  $y = h(t)$

the chain rule becomes

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$


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For  $w = f(x, y)$ ,  $x = g(s, t)$ ,  $y = h(s, t)$

the chain rule becomes

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}$$


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For  $z = f(x, y)$  the gradient of  $f$  is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \quad \text{or} \quad \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}$$

The directional derivative of  $f(x, y)$  in the direction of the unit vector  $\vec{u}$  is

$$D_{\vec{u}} f(x, y) = \nabla f \cdot \vec{u}$$

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A normal vector to the surface  $f(x, y, z) = 0$  at the point  $(x_0, y_0, z_0)$  is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

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The equation of the tangent plane to the surface  $f(x, y, z) = 0$  at the point  $(x_0, y_0, z_0)$  is

$$\nabla f \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0$$

$$\frac{\partial f}{\partial x} (x - x_0) + \frac{\partial f}{\partial y} (y - y_0) + \frac{\partial f}{\partial z} (z - z_0) = 0$$

The area of the surface  $z = f(x, y)$  over the region  $R$  is

$$\text{surface area} = \iint_R \sqrt{1 + f_x^2 + f_y^2} \, dA$$

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Definition of Conservative Vector Field.

A vector field  $F$  is called conservative if there is a scalar valued function  $f$  for which  $\nabla f = \vec{F}$ .

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Test for Conservative vector field.

$\vec{F}(x, y) = \langle M, N \rangle$  is conservative if and only if  $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

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The curl of  $\vec{F}(x, y, z) = \langle M, N, P \rangle$  is

defined as

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \vec{i} + \left( \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \vec{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \vec{k}$$

Test for Conservative Vector Field.

$\vec{F}(x, y, z) = \langle M, N, P \rangle$  is conservative

if and only if  
 $\text{curl } \vec{F} = \vec{0}$  i.e.,  $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$ ,  $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$ , and  $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

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The divergence of the vector field

$\vec{F}(x, y, z) = \langle M, N, P \rangle$  is

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

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For the smooth curve  $C$  given by

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \quad a \leq t \leq b$$

the line integral becomes

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

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For  $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

the arc length differential becomes

$$ds = \|\vec{r}'(t)\| dt \quad \text{i.e.,} \quad \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

For the smooth curve  $C$  given by  $\vec{r}(t)$ ,  $a \leq t \leq b$   
the line integral of the vector field  $\vec{F}$  in terms  
of  $\vec{r}$ ,  $s$ , and  $t$  is

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F} \cdot \vec{r}'(t) dt$$

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The Fundamental Theorem of Line Integrals  
for the conservative vector field  $F(x, y) = M\vec{i} + N\vec{j}$   
and the piecewise smooth curve  $C$  given by  
 $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$   $a \leq t \leq b$  states  
$$\int_C \vec{F} \cdot d\vec{r} = f(x(b), y(b)) - f(x(a), y(a))$$
 where  $\nabla f = \vec{F}$

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Let  $R$  be a simply connected region with a  
piecewise smooth boundary  $C$ , oriented clockwise,  
then Green's Theorem states

$$\int_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Let  $S$  be a smooth parametric surface

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

defined over  $D$  in the  $uv$ -plane. Then the

normal vector corresponding to  $(u_0, v_0)$  is

$$\vec{r}_u \times \vec{r}_v = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \times \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

For the smooth parametric surface given by

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

defined over  $D$  in the  $uv$ -plane, the surface area is given by

$$\iint_S dS = \iint_D \|\vec{r}_u \times \vec{r}_v\| dA$$

For the surface  $z = g(x,y)$  over  $R$  (in the

$xy$ -plane) the surface integral is given by

$$\iint_S f(x,y,z) dS = \iint_R f(x,y,g(x,y)) \sqrt{1 + g_x^2 + g_y^2} dA$$

**REVISED**

10:42 am, 5/28/06