

M252 Final Exam Practice for Chapters 10-13
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1. Find the magnitude of the vector \mathbf{v} given its initial and terminal points.

Initial point: $(4, -6, -3)$

$$\sqrt{(7-4)^2 + (-1-6)^2 + (-1-3)^2} = \sqrt{9+25+4} \\ = \sqrt{38}$$

Terminal point: $(7, -1, -1)$

2. Find the unit vector in the direction of \mathbf{u} .

$$\mathbf{u} = \langle 4, -3, -5 \rangle \quad \frac{\langle 4, -3, -5 \rangle}{\sqrt{4^2 + 3^2 + 5^2}} = \left\langle \frac{4}{5\sqrt{2}}, -\frac{3}{5\sqrt{2}}, -\frac{5}{5\sqrt{2}} \right\rangle$$

The possible solutions are given to two decimal places.

$$\left\langle \frac{2}{5}\sqrt{2}, -\frac{3\sqrt{2}}{10}, -\frac{\sqrt{2}}{2} \right\rangle$$

3. Find (a) $\mathbf{u} \cdot \mathbf{v}$ (b) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$ (c) $\mathbf{u} \cdot (4\mathbf{v})$ given the vectors \mathbf{u} and \mathbf{v} .

$$\mathbf{u} = \langle 1, 6 \rangle, \quad \mathbf{v} = \langle -3, -1 \rangle$$

$$(a) \mathbf{u} \cdot \mathbf{v}$$

$$1 \cdot (-3) + 6 \cdot (-1) \\ -3 - 6 = \boxed{-9}$$

$$(b) (\mathbf{u} \cdot \mathbf{v})\mathbf{v}$$

$$-9 \langle -3, -1 \rangle \\ \boxed{\langle 27, 9 \rangle}$$

$$(c) \mathbf{u} \cdot (4\mathbf{v})$$

$$\langle 1, 6 \rangle \cdot \langle -12, -4 \rangle \\ 1 \cdot (-12) + 6 \cdot (-4) \\ \boxed{-36}$$

4. Find the angle between the vectors for \mathbf{u} and \mathbf{v} given below.

$$\mathbf{u} = \langle 2, 6 \rangle, \quad \mathbf{v} = \langle -2, 3 \rangle$$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{2(-2) + 6 \cdot 3}{\sqrt{2^2 + 6^2} \cdot \sqrt{(-2)^2 + 3^2}} = \frac{14}{\sqrt{40} \sqrt{13}}$$

$$\theta = \arccos \left(\frac{7}{\sqrt{130}} \right) \approx .90975$$

5. Determine whether \mathbf{u} and \mathbf{v} are orthogonal, parallel, or neither.

$$\mathbf{u} = \langle 4, -5 \rangle, \quad \mathbf{v} = \langle 4, -5 \rangle$$

$$\mathbf{u} \cdot \mathbf{v} = 4 \cdot 4 + (-5)(-5) = 16 + 25 = 41 \Rightarrow \text{NOT orthogonal}$$

$$\mathbf{u} = 1 \cdot \mathbf{v} \Rightarrow \text{parallel}$$

6. Find the direction cosines of the vector \mathbf{u} given below.

$$\mathbf{u} = \langle 2, 5, -2 \rangle$$

$$\|\mathbf{u}\| = \sqrt{2^2 + 5^2 + 2^2} = \sqrt{33}$$

$$\cos \alpha = \frac{2}{\|\mathbf{u}\|} = \frac{2}{\sqrt{33}}$$

$$\cos \beta = \frac{5}{\|\mathbf{u}\|} = \frac{5}{\sqrt{33}}$$

$$\cos \gamma = \frac{-2}{\|\mathbf{u}\|} = -\frac{2}{\sqrt{33}}$$

7. Find the projection of \mathbf{u} onto \mathbf{v} , and the vector component of \mathbf{u} orthogonal to \mathbf{v} .

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Projection of \mathbf{u} onto \mathbf{v}

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} \hat{\mathbf{v}} = \frac{1 \cdot 0 + 9 \cdot 3}{0 + 3 \cdot 3} \langle 0, 3 \rangle = \langle 0, 9 \rangle$$

Component of \mathbf{u} orthogonal to \mathbf{v}

$$\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u} = \langle 1, 9 \rangle - \langle 0, 9 \rangle = \langle 1, 0 \rangle$$

8. Given the vectors \mathbf{u} and \mathbf{v} , find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{v}$.

$$\mathbf{u} = \langle 1, 9, 2 \rangle, \quad \mathbf{v} = \langle -1, 2, 3 \rangle$$

$$\begin{array}{c} \mathbf{u} \times \mathbf{v} \\ \left| \begin{array}{ccc} i & j & k \\ 1 & 9 & 2 \\ -1 & 2 & 3 \end{array} \right| = (27 - 4) \mathbf{i} + (-2 - 3) \mathbf{j} + (2 + 9) \mathbf{k} \\ = \langle 23, -5, 11 \rangle \end{array} \quad \begin{array}{c} \mathbf{v} \times \mathbf{v} \\ \left| \begin{array}{ccc} i & j & k \\ -1 & 2 & 3 \\ -1 & 2 & 3 \end{array} \right| = \langle 0, 0, 0 \rangle \end{array}$$

9. Find a set of parametric equations of the line through the point $(-5, 5, 2)$ parallel to the vector $\mathbf{v} = \langle 7, 5, 4 \rangle$.

$$\vec{r}(t) = \langle -5, 5, 2 \rangle + t \langle 7, 5, 4 \rangle$$

$$x = 7t - 5$$

$$y = 5t + 5$$

$$z = 4t + 2$$

10. Find an equation of a plane passing through the point given and perpendicular to the given vector.

Point: $(6, 8, 7)$ Vector $\mathbf{v} = \langle 7, 8, 2 \rangle$

$$\langle 7, 8, 2 \rangle \cdot (\langle x, y, z \rangle - \langle 6, 8, 7 \rangle) = 0$$

$$7(x-6) + 8(y-8) + 2(z-7) = 0$$

11. Find the distance between the point $(1, -2, -1)$ and the plane given below.

$-5x - 10y + 5z = 15$ use magnitude of projection of any vector to the plane onto the normal vector: $(0, 0, 3)$ is on the plane,

$$\langle 0, 0, 3 \rangle - \langle 1, -2, -1 \rangle = \langle -1, 2, 4 \rangle$$
 goes to plane,

$$\text{proj}_{\langle -5, -10, 5 \rangle} \langle -1, 2, 4 \rangle = \frac{|\langle -1, 2, 4 \rangle \cdot \langle -5, -10, 5 \rangle|}{\sqrt{(-5)^2 + (-10)^2 + 5^2}} = \frac{5 - 20 + 20}{\sqrt{145}} = \frac{5}{\sqrt{145}}$$

12. Find an equation in cylindrical coordinates for the equation given in rectangular coordinates.

$$9x^2 + 9y^2 = 2x \quad 9r^2 = 2r \cos \theta$$

$$\Rightarrow r = \frac{2}{9} \cos \theta$$

13. Find an equation in rectangular coordinates for the equation given in spherical coordinates.

$$\rho = 4 \csc \phi \sec \theta$$

$$\Rightarrow \csc \phi = \frac{\rho}{r}$$

$$\sec \theta = \frac{\rho}{x}$$

$$\rho = 4 \frac{r}{\csc \phi} \frac{r}{\sec \theta} \Rightarrow 1 = \frac{4}{x} \Rightarrow x = 4$$

14. Convert the following point from cylindrical coordinates to spherical coordinates.

$$\left(8, \frac{\pi}{6}, 8\right)$$

$$\Rightarrow \phi = \frac{\pi}{4} \quad \rho = 8\sqrt{2}, \text{ and of course } \theta = \frac{\pi}{6}$$

$$\left\langle 8\sqrt{2}, \frac{\pi}{6}, \frac{\pi}{4} \right\rangle$$

15. Represent the following curve by a vector valued function.

$$\frac{x^2}{49} + \frac{y^2}{16} = 1, x > 0 \quad r(t) = 7\cos(t)\vec{i} + 4\sin(t)\vec{j}$$

$$\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

16. Find a vector-valued function, using the given parameter, to represent the intersection of the surfaces given below.

Surfaces

$$z = \frac{x^2}{81} + \frac{y^2}{4}, y + 11x = 0$$

Parameter

$$x = t$$

$$y = -11t$$

$$z = \frac{t^2}{81} + \frac{121t^2}{4} = \left(\frac{1}{81} + \frac{121}{4}\right)t^2$$

17. The position vector \mathbf{r} describes the path of an object moving in the xy -plane. Find the velocity and acceleration vectors at the given point.

$$\mathbf{r}(t) = -4t^4\vec{i} + 5t^4\vec{j}, \quad (-324, 405) \implies t = \sqrt[4]{81} = 3$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = -16t^3\vec{i} + 20t^3\vec{j}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = -48t^2\vec{i} + 60t^2\vec{j}$$

$$\mathbf{v}(3) = -432\vec{i} + 540\vec{j}$$

$$\mathbf{a}(3) = -432\vec{i} + 540\vec{j}$$

18. The position vector \mathbf{r} describes the path of an object moving in space. Find the velocity, speed, and acceleration of the object.

$$\mathbf{r}(t) = 4\vec{i} + 3t^2\vec{j} - 8t^2\vec{k}$$

Velocity

$$\mathbf{v}(t) = \mathbf{r}'(t) = 6t\vec{j} - 16t\vec{k}$$

Speed

$$s = \|\mathbf{r}'(t)\|$$

$$= \sqrt{(6t)^2 + (-16t)^2} = 2\sqrt{73}t$$

Acceleration

$$\mathbf{a}(t) = \mathbf{r}''(t)$$

$$= 6\vec{j} - 16\vec{k}$$

19. Use the given acceleration function to find the velocity and position vector. Then find the position at time $t = 3$.

$$\mathbf{a}(t) = 10\vec{i} + 8\vec{j} + 8\vec{k}, \quad \mathbf{v}(0) = 3\vec{k}, \quad \mathbf{r}(0) = \mathbf{0}$$

$$\mathbf{r}(3) = 45\vec{i} + 36\vec{j} + 45\vec{k}$$

$$\mathbf{v}(t) = 10t\vec{i} + 8t\vec{j} + (8t+3)\vec{k}$$

$$\mathbf{r}(t) = 5t^2\vec{i} + 4t^2\vec{j} + (4t^2 + 3t)\vec{k}$$

20. Find the unit tangent vector $\mathbf{T}(t)$ and then use it to find a set of parametric equations for the line tangent to the space curve given below at the given point.

$$\mathbf{r}(t) = -5t\mathbf{i} - 4t^2\mathbf{j} + 5t\mathbf{k}, \quad t=5$$

$$\|\mathbf{r}'(5)\| = \sqrt{(-5)^2 + (-40)^2 + 5^2} = 5\sqrt{66}$$

$$\mathbf{r}'(t) = -5\mathbf{i} - 8t\mathbf{j} + 5\mathbf{k}$$

$$\mathbf{T} = \frac{\mathbf{r}'(5)}{\|\mathbf{r}'(5)\|} = \frac{<-1, -8, 1>}{\sqrt{66}}$$

$$\begin{aligned} x &= -25 - t \\ y &= -20 - 8t \\ z &= 25 + t \end{aligned}$$

21. Find the principle unit normal vector to the curve given below at the specified point.

$$\mathbf{r}(t) = t\mathbf{i} + 6t^2\mathbf{j}, \quad t=1$$

$$\mathbf{T}(1) = \frac{\mathbf{i} + 12\mathbf{j}}{\sqrt{145}} \Rightarrow \mathbf{N}(1) = \frac{-12\mathbf{i} + \mathbf{j}}{\sqrt{145}}$$

$$\mathbf{r}'(t) = \mathbf{i} + 12t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1^2 + (12t)^2} = \sqrt{1+144t^2}$$

$$\mathbf{T}' = -f(t)(\mathbf{i} + 12t\mathbf{j}) + \frac{12\mathbf{j}}{\sqrt{1+144t^2}} \Rightarrow \text{quadrant II}$$

22. Find the length of the plane curve given below.

$$\mathbf{r}(t) = 4\mathbf{i} + 3t^2\mathbf{j}, \quad [0, 6]$$

$$\mathbf{r}'(t) = 6t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 6|t|$$

$$\text{arc length} = \int_0^6 \|\mathbf{r}'(t)\| dt = \int_0^6 6t dt = [3t^2]_0^6 = 108$$

23. Find the length of the space curve given below.

$$\mathbf{r}(t) = 3\mathbf{i} + 4\cos t\mathbf{j} + 4\sin t\mathbf{k}, \quad [0, 3]$$

$$\mathbf{r}'(t) = -4\sin t\mathbf{j} + 4\cos t\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-4\sin t)^2 + (4\cos t)^2} = 4$$

$$\int_0^3 4dt = [4t]_0^3 = 12$$

24. Find the four second partial derivatives. Observe that the second mixed partials are equal.

$$z = x^2 + 9xy + 7y^2$$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial^2 z}{\partial y^2} = 14$$

$$\frac{\partial z}{\partial x} = 2x + 9y$$

$$\frac{\partial^2 z}{\partial y \partial x} = 9$$

$$\frac{\partial^2 z}{\partial x \partial y} = 9$$

$$\frac{\partial z}{\partial y} = 9x + 14y$$

25. Find the total differential of the function $z = \frac{x^8}{y}$.

$$\frac{\partial z}{\partial x} = \frac{8x^7}{y}$$

$$\frac{\partial z}{\partial y} = -\frac{x^8}{y^2}$$

$$dz = \frac{8x^7}{y} dx - \frac{x^8}{y^2} dy$$

26. Let $w = xy$, where $x = 2 \sin t$ and $y = -7 \cos t$. Find $\frac{dw}{dt}$.

$$\begin{aligned}\frac{\partial w}{\partial x} &= y & \frac{dx}{dt} &= 2 \cos t & \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ \frac{\partial w}{\partial y} &= x & \frac{dy}{dt} &= 7 \sin t & &= 2y \cos t + 7x \sin t\end{aligned}$$

$$\begin{aligned}-14 \cos^2 t + 14 \sin^2 t \\ = -14 \cos(2t)\end{aligned}$$

27. Let $w = x^4 + y^4$, where $x = 3s + t$, $y = 3s - t$. Find $\frac{\partial w}{ds}$ and $\frac{\partial w}{dt}$ and evaluate each

partial derivative at the point $s = -2, t = 2$.

$$\begin{aligned}\frac{\partial w}{\partial x} &= 4x^3 & \frac{\partial x}{\partial s} &= 3 & \frac{\partial x}{\partial t} &= 1 & \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} = 4x^3 \cdot 3 + 4y^3 \cdot 3 = 12(3s+t)^3 + 12(3s-t)^3 \\ \frac{\partial w}{\partial y} &= 4y^3 & \frac{\partial y}{\partial s} &= 3 & \frac{\partial y}{\partial t} &= -1 & \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} = 4x^3 - 4y^3 = 4(3s+t)^3 - 4(3s-t)^3\end{aligned}$$

28. Find the directional derivative of the function at P in the direction of \vec{v} .

$$f(x, y) = 10x - 10xy + 9y, \quad P(1, 6), \quad \vec{v} = \frac{1}{2}(\hat{i} + \sqrt{3}\hat{j})$$

$$\begin{aligned}\nabla f &= \langle 10 - 10y, -10x + 9 \rangle \\ \text{at } (1, 6) \\ \nabla f &= \langle -50, -1 \rangle\end{aligned}$$

$$\begin{aligned}D_{\vec{v}} &= \nabla f \cdot \vec{v} = \langle -50, -1 \rangle \cdot \frac{1}{2} \langle 1, \sqrt{3} \rangle \\ &= -\frac{50 + \sqrt{3}}{2}\end{aligned}$$

29. Find the gradient of the function at the given point.

$$f(x, y) = 6x - 8y^2 + 7, \quad (6, 1)$$

$$\begin{aligned}\nabla f &= \langle 6, -16y \rangle \\ \text{at } (6, 1) \\ \nabla f &= \langle 6, -16 \rangle\end{aligned}$$

30. Find a unit normal vector to the surface $x + y + z = 6$ at the point $(3, 0, 3)$.

$$\begin{aligned}\nabla(x + y + z - 6) &= \langle 1, 1, 1 \rangle \text{ a normal vector} \\ \text{unit normal is } &\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle\end{aligned}$$

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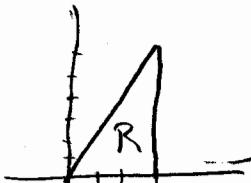
31. Find a unit normal vector to the surface $x^2 + y^2 + z^2 = 11$ at the point $(3, 1, 1)$.

$$\begin{aligned}\nabla(x^2 + y^2 + z^2 - 11) &= \langle 2x, 2y, 2z \rangle \\ \text{at } (3, 1, 1) \text{ normal vector is } &\langle 6, 2, 2 \rangle \\ \text{unit normal vector is } &\left\langle \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}} \right\rangle\end{aligned}$$

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32. Find an equation of the tangent plane to the surface $g(x, y) = x^2 - y^2$ at the point $(2, 7, -45)$. normal vector is $\nabla(z - g(x, y)) = \langle -2x, 2y, 1 \rangle$
at $(2, 7, -45)$ $\vec{n} = \langle -4, 14, 1 \rangle$
so tangent plane is $\langle -4, 14, 1 \rangle \cdot (\langle x, y, z \rangle - \langle 2, 7, -45 \rangle) = 0$
 $\Rightarrow -4(x-2) + 14(y-7) + (z+45) = 0$
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33. Find an equation of the tangent plane to the surface $x^2 + 5y^2 + z^2 = 160$ at the point $(4, -4, 8)$. $\vec{n} = \nabla(x^2 + 5y^2 + z^2 - 160) = \langle 2x, 10y, 2z \rangle$
at $(4, -4, 8)$ $\vec{n} = \langle 8, -40, 16 \rangle$ or $\langle 1, -5, 2 \rangle$
tangent plane $\langle 1, -5, 2 \rangle \cdot (\langle x, y, z \rangle - \langle 4, -4, 8 \rangle) = 0$ $\Rightarrow (x-4) - 5(y+4) + 2(z-8) = 0$,
or $x - 5y + 2z = 40$
34. Find an equation of the tangent plane and find symmetric equations of the normal line to the surface $xyz = 8$ at the point $(1, 4, 2)$.
 $\vec{n} = \nabla(xyz - 8) = \langle yz, xz, xy \rangle$ at $(1, 4, 2) = \langle 8, 2, 4 \rangle$ or $\langle 4, 1, 2 \rangle$
tangent plane $\langle 4, 1, 2 \rangle \cdot (\langle x, y, z \rangle - \langle 1, 4, 2 \rangle) = 0$ $4(x-1) + (y-4) + 2(z-2) = 0$
normal line $\langle 1, 4, 2 \rangle + t\langle 4, 1, 2 \rangle \Rightarrow x = 4t + 1, y = t + 4, z = 2t + 2$
35. Sketch the region R and evaluate the iterated integral.

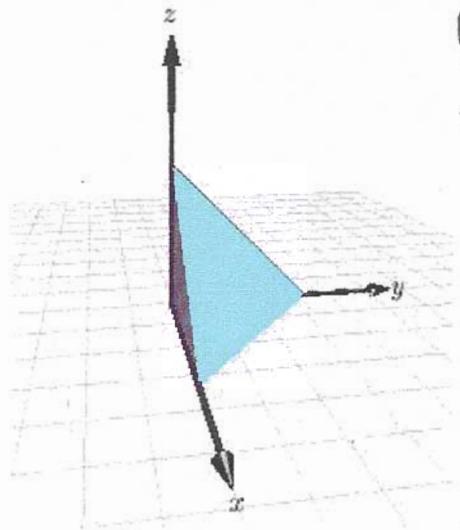
$$I = \int_0^{3/2x} \int_0^{2x} (x+y) dy dx$$



$$I = \int_0^{3/2x} \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=2x} dx =$$

$$\int_0^{3/2x} 2x^2 + 2x^2 dx = \left[\frac{4x^3}{3} \right]_0^{3/2x} = 36$$

36. Use a double integral to find the volume of the indicated solid.



$$\int_0^{\frac{5}{4}} \int_0^{\frac{5-4x}{4}} \frac{5-4x-4y}{4} dy dx$$

$$\int_0^{\frac{5}{4}} \left[\frac{5y - 4xy - 2y^2}{4} \right]_{y=0}^{y=\frac{5-4x}{4}} dx$$

$$\int_0^{\frac{5}{4}} \frac{5\left(\frac{5-4x}{4}\right) - 4x\left(\frac{5-4x}{4}\right) - 2\left(\frac{5-4x}{4}\right)^2}{4} dx$$

$$\frac{1}{32} \int_0^{\frac{5}{4}} 25 - 40x + 16x^2 dx$$

$$\frac{1}{32} \left[25x - 20x^2 + \frac{16x^3}{3} \right]_0^{\frac{5}{4}} = \frac{125}{384} = \frac{1}{6} \left(\frac{5}{4}\right)^3$$

37. Set up a double integral to find the volume of the solid bounded by the graphs of the equations given below.

$$\int_0^{\frac{4}{3}} \int_{y=3x}^{y=4} xy^3 dy dx$$

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38. Use a double integral in polar coordinates to find the volume of the solid in the first octant bounded by the graphs of the equations given below.

$$z = x^3 y, \quad x^2 + y^2 = 25 \quad 0 \leq r \leq 5 \quad 0 \leq \theta \leq \frac{\pi}{2} \quad 0 \leq z \leq x^3 y$$

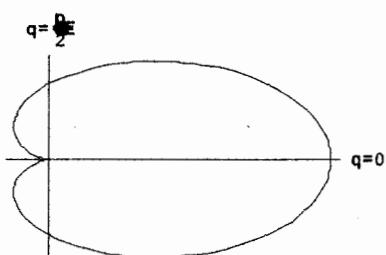
$$x^3 y = r^4 \cos^3 \theta \sin \theta$$

$$\int_0^{\frac{\pi}{2}} \int_0^5 r^5 \cos^3 \theta \sin \theta dr d\theta = \int_0^5 \left[\frac{r^6}{6} \right]_0^5 \cos^3 \theta \sin \theta d\theta$$

$$= \frac{5^6}{6} \left[-\frac{u^4}{4} \right]_1^0 = \frac{5^6}{24}$$

$u = \cos \theta$
 $du = -\sin \theta d\theta$

39. Use a double integral to find the area enclosed by the graph of $r = 3 + 3\cos\theta$.



$$\begin{aligned}
 & 2 \int_0^{\pi} \int_0^{r=3+3\cos\theta} r dr d\theta = 2 \left[\frac{r^2}{2} \right]_0^{\pi} \int_0^{3+3\cos\theta} d\theta \\
 & = \int_0^{\pi} (3+3\cos\theta)^2 d\theta = 9 \left[\theta + 2\sin\theta + \frac{\theta - \sin\theta\cos\theta}{2} \right]_0^{\pi} \\
 & = 9 \left(\pi + \frac{\pi}{2} \right) = \frac{27\pi}{2}
 \end{aligned}$$

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40. Find the area of the surface given by $z = f(x, y)$ over the region R .

$$f(x, y) = -6 - 5x + 4y \quad \int_0^2 \int_0^2 \sqrt{1 + f_x^2 + f_y^2} dx dy$$

R : square with vertices $(0, 0), (2, 0), (2, 2), (0, 2)$

$$= \int_0^2 \int_0^2 \sqrt{1 + 25 + 16} dx dy = \sqrt{42} \cdot 4 = 4\sqrt{42}$$

41. Find the area of the surface given by $z = f(x, y)$ over the region R .

$$f(x, y) = 1 - x^2 \quad \int_0^1 \int_0^1 \sqrt{1 + f_x^2 + f_y^2} dx dy$$

R : rectangle with vertices $(0, 0), (1, 0), (1, 1), (0, 1)$

$$\begin{aligned}
 & = \int_0^1 \int_0^1 \sqrt{1 + 4x^2} dx dy = 2 \left[\frac{1}{2} \left(x \sqrt{x^2 + \frac{1}{4}} + \frac{1}{4} \ln|x + \sqrt{x^2 + \frac{1}{4}}| \right) \right]_0^1 \\
 & \quad \sqrt{\frac{5}{4}} + \frac{1}{4} \ln \left(1 + \sqrt{\frac{5}{4}} \right) - \frac{1}{4} \ln \frac{1}{2} = \frac{1}{4} (2\sqrt{5} + \ln(2 + \sqrt{5}))
 \end{aligned}$$

42. Find the area of the surface given by $z = f(x, y)$ over the region R .

$$f(x, y) = xy \quad \int_R \int \sqrt{1 + f_x^2 + f_y^2} dA = \int_R \int \sqrt{1 + x^2 + y^2} dA$$

$R: \{(x, y) : x^2 + y^2 \leq 9\}$

in polar coordinates we have

$$\begin{aligned}
 & = \int_0^{2\pi} \int_0^3 r \sqrt{1 + r^2} dr d\theta = \frac{2\pi}{2} \int_1^{10} \sqrt{u} du = \pi \left[\frac{2}{3} u^{\frac{3}{2}} \right]_1^{10} \\
 & \quad u = r^2 \quad du = 2r dr \\
 & \quad = \frac{2\pi}{3} (10\sqrt{10} - 1)
 \end{aligned}$$

$$du = 2r dr$$

43. Evaluate the following iterated integral.

$$\begin{aligned} \int_0^2 \int_0^1 \int_0^1 (-4x+5y+z) dx dy dz &= \int_0^2 \int_0^1 \left[-2x^2 + 5xy + xz \right]_0^1 dy dz \\ &= \int_0^2 \int_0^1 -2 + 5y + z dy dz = \int_0^2 \left[-2y + \frac{5y^2}{2} + yz \right]_0^1 dz = \int_0^2 \frac{1}{2} + z dz \\ &= \left[\frac{z}{2} + \frac{z^2}{2} \right]_0^2 = \boxed{3} \end{aligned}$$

44. Convert the integral below from rectangular coordinates to both cylindrical and spherical coordinates, and evaluate the simpler iterated integral.

$$\int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} \sqrt{x^2+y^2+z^2} dz dy dx$$

The region of this

integral is the first quadrant of the sphere of radius 5. The integrand is ρ .

Spherical coordinates

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^5 \rho^3 \sin\phi d\rho d\phi d\theta = \frac{\pi}{2} \left[\frac{\rho^4}{4} \right]_0^5 \int_0^{\frac{\pi}{2}} \sin\phi d\phi$$

$$= \boxed{\frac{5^4 \pi}{8}}$$

Cylindrical coordinates

$$\int_0^{\frac{\pi}{2}} \int_0^5 \int_0^{\sqrt{25-r^2}} r \sqrt{r^2+z^2} dz dr d\theta$$

(more difficult)