1. Find the gradient vector for the scalar function. (That is, find the conservative vector field for the potential function.)

$f(x,y) = 7x^2 + 8xy + 2y^2$

$\nabla f = \langle 14x+8y, 8x+4y \rangle$
or $(14x+8y)\vec{i} + (8x+4y)\vec{j}$

2. Find the gradient vector for the scalar function. (That is, find the conservative vector field for the potential function.)

$f(x,y) = \sin 9x \cos 3y$

$\nabla f = \langle 9\cos 9x \cos 3y, -3\sin 9x \sin 3y \rangle$

3. Determine whether the vector field is conservative.

$\vec{\mathbf{F}}(x,y) = 9y^2\left(7y\hat{\mathbf{i}} - x\hat{\mathbf{j}}\right)$

$N = -9xy^2 \quad M = 63y^3$

$\frac{\partial N}{\partial x} = -9y^2 \quad \frac{\partial M}{\partial y} = 189y^2$

A) Conservative
B) Not Conservative

$\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$

4. Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

$\vec{\mathbf{F}}(x,y) = 7x^6y\hat{\mathbf{i}} + x^7\hat{\mathbf{j}}$

$\frac{\partial\, 7x^6y}{\partial y} = 7x^6 \quad \frac{\partial\, x^7}{\partial x} = 7x^6 \Rightarrow$ conservative

$f(x,y) = x^7y + c$

5. Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

$\vec{\mathbf{F}}(x,y) = \frac{6y}{x}\hat{\mathbf{i}} - \frac{x^6}{y^6}\hat{\mathbf{j}}$

$\frac{\partial \frac{6y}{x}}{\partial y} = \frac{6}{x} \quad \frac{\partial \frac{-x^6}{y^6}}{\partial x} = -\frac{6x^5}{y^6} \Rightarrow$ NOT conservative

6. Find the curl for the vector field at the given point.

$\vec{\mathbf{F}}(x,y,z) = 2xyz\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}, \quad (2,3,2)$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & 2y & 2z \end{vmatrix} = 0\vec{i} + 2xy\vec{j} - 2xz\vec{k}$$

at $(2,3,2)$ we have
curl $F = \langle 0, 12, -8 \rangle$

7. Determine whether the vector field is conservative. If it is, find a potential function for the vector field. $\frac{\partial M}{\partial y} = 12x^2y^3z^5 = \frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial z} = 15x^2y^4z^4 = \frac{\partial P}{\partial x}$,

$$\vec{\mathbf{F}}(x,y,z) = 3x^2y^4z^5\hat{\mathbf{i}} + 4x^3y^3z^5\hat{\mathbf{j}} + 5x^3y^4z^4\hat{\mathbf{k}}$$ $\frac{\partial N}{\partial z} = 20x^3y^3z^4 = \frac{\partial P}{\partial y}$

$f(x,y,z) = x^3y^4z^5$

8. Find the divergence of the vector field.

$\vec{\mathbf{F}}(x,y,z) = 9x^4\hat{\mathbf{i}} - xy^3\hat{\mathbf{j}}$ $\frac{\partial 9x^4}{\partial x} + \frac{\partial -xy^3}{\partial y} = 36x^3 - 3xy^2$

9. Find the divergence of the vector field at the given point.

$\vec{\mathbf{F}}(x,y,z) = 8xyz\hat{\mathbf{i}} + 8y\hat{\mathbf{j}} + 8x\hat{\mathbf{k}}$, $(8,9,8)$ $\frac{\partial 8xyz}{\partial x} + \frac{\partial 8y}{\partial y} + \frac{\partial 8x}{\partial z} = 8yz + 8$

at $(8,9,8)$, $\nabla \cdot F = 8 \cdot 9 \cdot 8 + 8 = 584$

12. Evaluate the line integral along the given path.

$\int_C (2x - 7y)\,ds \quad C: \vec{\mathbf{r}}(t) = 7t\hat{\mathbf{i}} + 3t\hat{\mathbf{j}}, \quad 0 \le t \le 8$

$r'(t) = \langle 7, 3 \rangle \quad \|r'(t)\| = \sqrt{49+9} = \sqrt{58} \quad ds = \|r'(t)\|\,dt = \sqrt{58}\,dt$

$$\int_C (2x-7y)\,ds = \int_0^8 \left(2(7t) - 7(3t)\right)\sqrt{58}\,dt$$

$$\sqrt{58}\int_0^8 -7t\,dt = \sqrt{58}\left(-7\cdot\frac{8^2}{2}\right) = -224\sqrt{58}$$

13. Evaluate $\int_C (x^2+y^2)\,ds$ where the path $C$ is:

(i) the $x$-axis from $x=0$ to $x=1$; $x=t \quad y=0 \quad r(t)=\langle t,0\rangle \quad r'(t)=\langle 1,0\rangle$

$ds=\|r'(t)\|dt=dt$ $\quad \int_0^1 t^2+0^2\,dt=\left[\frac{t^3}{3}\right]_0^1=\frac{1}{3}$

(ii) the $y$-axis from $y=1$ to $y=3$. $x=0,\ y=t,\ 1\le t\le 3,\ \|r'(t)\|=1 \quad ds=dt$

$\int_1^3 0^2+t^2\,dt=\left[\frac{t^3}{3}\right]_1^3=9-\frac{1}{3}=\frac{26}{3}$

14. Evaluate $\int_C (x+49\sqrt{y})\,ds$ where the path $C$ is:

(i) the line from (0,0) to (1,1); $r(t)=\langle t^2,t^2\rangle,\ 0\le t\le 1 \quad r'(t)=\langle 2t,2t\rangle, \quad ds=2\sqrt{2}\,t\,dt$

$\int_0^1 (t^2+49t)2\sqrt{2}\,t\,dt=2\sqrt{2}\int_0^1 t^3+49t^2\,dt=2\sqrt{2}\left[\frac{t^4}{4}+\frac{49t^3}{3}\right]_0^1=2\sqrt{2}\left(\frac{1}{4}+\frac{49}{3}\right)$

$=199\sqrt{2}/6$

(ii) the line from (0,0) to (3,9). $r(t)=\langle 3t^2,9t^2\rangle \quad 0\le t\le 1 \quad r'(t)=\langle 6t,18t\rangle$

$ds=6t\sqrt{10}\,dt$

$\int_0^1 (3t^2+3\cdot 49t)6t\sqrt{10}\,dt=18\sqrt{10}\int_0^1 t^3+49t^2\,dt=18\sqrt{10}\left[\frac{t^4}{4}+\frac{49t^3}{3}\right]_0^1=18\sqrt{10}\left(\frac{1}{4}+\frac{49}{3}\right)$

$=\frac{597\sqrt{10}}{2}$

15. Evaluate $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$ where C is represented by $\hat{\mathbf{r}}(t)$.

$\vec{\mathbf{F}}(x,y)=xy\hat{\mathbf{i}}+y\hat{\mathbf{j}}$

$C:\vec{\mathbf{r}}(t)=32t\hat{\mathbf{i}}+t\hat{\mathbf{j}},\quad 0\le t\le 8$

$d\vec{r}=\langle 32,1\rangle dt$

$\int_0^8 \langle xy,y\rangle\cdot\langle 32,1\rangle dt$

$=\int_0^8 \langle 32t\cdot t,t\rangle\langle 32,1\rangle dt \rightarrow \int_0^8 (32^2t^2+t)dt$

$=\frac{32^2\cdot 8^3}{3}+\frac{8^2}{2}$

$=174794.\frac{2}{3}$

16. Evaluate $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$ where C is represented by $\vec{\mathbf{r}}(t)$.

$\int_{-2}^{2} \langle 5t, 9\sqrt{4-t^2}\rangle\cdot\langle 1,\frac{-t}{\sqrt{4-t^2}}\rangle dt$

$\vec{\mathbf{F}}(x,y)=5x\hat{\mathbf{i}}+9y\hat{\mathbf{j}}$

$=\int_{-2}^{2}(5t-9t)dt=\left[-2t^2\right]_{-2}^{2}=0$

$C:\vec{\mathbf{r}}(t)=t\hat{\mathbf{i}}+\sqrt{4-t^2}\hat{\mathbf{j}},\quad -2\le t\le 2$

$d\vec{r}=\langle 1,\frac{-t}{\sqrt{4-t^2}}\rangle dt$

17. Evaluate the line integral along the path $C$ given by $x=4t$, $y=20t$, where $0\le t\le 1$.

$\int_C (x+4y^2)\,ds$ $\quad \vec{r}(t)=4t\langle 1,5\rangle \quad 0\le t\le 1 \quad r'(t)=4\langle 1,5\rangle$

$ds=4\sqrt{26}\,dt$

$\int_0^1 (4t+4(20t)^2)4\sqrt{26}\,dt=16\sqrt{26}\int_0^1 t+400t^2\,dt=16\sqrt{26}\left[\frac{t^2}{2}+\frac{400t^3}{3}\right]_0^1$

$=16\sqrt{26}\left(\frac{1}{2}+\frac{400}{3}\right)=\frac{6424}{3}\sqrt{26}$

18. Evaluate the line integral

$$\int_C (2x-y)\,dx+(x+3y)\,dy$$

along the path $C$, where $C$ is:

(i) the $x$-axis from $x=0$ to $x=8$. $r(t)=\langle t,0\rangle,\ dx=dt\ \ dy=0$

$$\int_0^8 (2t-0)\,dt+(t+3\cdot 0)\cdot 0=\left[t^2\right]_0^8=64$$

(ii) the $y$-axis from $y=0$ to $y=12$. $r(t)=\langle 0,t\rangle,\ dx=0,\ dy=dt$

$$\int_0^{12} (2\cdot 0-t)\cdot 0+(0+3t)\,dt=\left[\frac{3t^2}{2}\right]_0^{12}=216$$

19. Find the area of the lateral surface over the curve $C$ in the $xy$-plane and under the surface $z=f(x,y)$, where

Lateral surface area $=\int_C f(x,y)\,ds$

$f(x,y)=4$, C: line from (0,0) to (5,6).

$r(t)=\langle 5t,6t\rangle\quad ds=\sqrt{25+36}\,dt$

$$\int_0^1 4\sqrt{61}\,dt=4\sqrt{61}$$

20. Set up and evaluate the integral $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$ for each parametric representation of $C$.

$\vec{\mathbf{F}}(x,y)=x^2\hat{\mathbf{i}}+xy\hat{\mathbf{j}}$ on C $\vec{F}=\langle 9t^2,21t^3\rangle$

(i) $\vec{\mathbf{r}}_1(t)=3t\hat{\mathbf{i}}+7t^2\hat{\mathbf{j}},\quad 0\le t\le 1$ $d\vec{r}=\langle 3,14t\rangle dt$

$$\int_0^1 3t^2\langle 3,7t\rangle\cdot\langle 3,14t\rangle\,dt$$
$$=3\int_0^1 9t^2+98t^4\,dt$$
$$=3\left[3t^3+\frac{98}{5}t^5\right]_0^1$$
$$=3\left(3+\frac{98}{5}\right)=67.8$$

(ii) $\vec{\mathbf{r}}_2(t)=3\sin\theta\hat{\mathbf{i}}+7\sin^2\theta\hat{\mathbf{j}},\quad 0\le\theta\le\frac{\pi}{2}$

on C $\vec{F}=\langle 9\sin^2\theta,21\sin^3\theta\rangle$

$d\vec{r}=\langle 3\cos\theta,14\sin\theta\cos\theta\rangle d\theta$

$$\int_0^{\frac{\pi}{2}} 3\sin^2\theta\cos\theta\langle 3,7\sin\theta\rangle\cdot\langle 3,14\sin\theta\rangle\,d\theta$$

$u=\sin\theta$
$du=\cos\theta\,d\theta$

$$=3\int_0^{\frac{\pi}{2}}\sin^2\theta\cos\theta\left(9+98\sin^2\theta\right)d\theta$$

$$3\int_0^1 u^2(9+98u^2)\,du=3\left[3u^3+\frac{98}{5}u^5\right]_0^1=3\left(3+\frac{98}{5}\right)$$

$$=67.8$$

21. Determine whether or not the vector field is conservative.

$\vec{\mathbf{F}}(x,y)=40x^7y^7\hat{\mathbf{i}}+35x^8y^6\hat{\mathbf{j}}$

A) Conservative

B) Not conservative

$$\frac{\partial M}{\partial y} = 280x^7y^6 \qquad \frac{\partial N}{\partial x} = 280x^7y^6$$

22. Find the value of the line integral $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y)=2xy\hat{\mathbf{i}}+x^2\hat{\mathbf{j}}$.

(i) $\vec{\mathbf{r}}_1(t)=t\hat{\mathbf{i}}+t^3\hat{\mathbf{j}},\quad 0\le t\le 1$ $\qquad \vec{F}=\langle 2t^4, t^2\rangle,\ d\vec{r}=\langle 1, 3t^2\rangle dt$

$$\int_0^1 t^2\langle 2t^2, 1\rangle\cdot\langle 1, 3t^2\rangle dt = \int_0^1 5t^4\,dt = \left[t^5\right]_0^1 = 1$$

(ii) $\vec{\mathbf{r}}_2(t)=t\hat{\mathbf{i}}+t^7\hat{\mathbf{j}},\quad 0\le t\le 1$ $\qquad \vec{F}=\langle 2t^8, t^2\rangle,\ d\vec{F}=\langle 1, 7t^6\rangle dt$

$$\int_0^1 t^2\langle 2t^6, 1\rangle\cdot\langle 1, 7t^6\rangle dt = \int_0^1 9t^8\,dt = \left[t^9\right]_0^1 = 1$$

23. Find the value of the line integral $\int_C (2x-5y+7)dx-(5x+7y-8)dy$.

(i) $C$: the curve $x=\sqrt{49-y^2}$ from $(0,-7)$ to $(0,7)$

$x=7\cos\theta,\ y=7\sin\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$dx=-7\sin\theta\,d\theta \quad dy=7\cos\theta\,d\theta$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2(7\cos\theta)-5(7\sin\theta)+7)(-7\sin\theta\,d\theta)-(5(7\cos\theta)+7(7\sin\theta)-8)(7\cos\theta\,d\theta)$$

factor out 7 and drop odd terms

$$7\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(35(\sin^2\theta-\cos^2\theta)+8\cos\theta\right)d\theta = 7\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -35\cos(2\theta)+8\cos\theta\,d\theta = 7\left[-\frac{35}{2}\sin(2\theta)+8\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 112$$

(ii) $C$: from $(0,-7)$ to $(0,7)$ along $x=\sqrt{49-y^2}$, then back to $(0,-7)$ along the $y$-axis (a closed curve).

Using Green's Theorem $\iint_D (-5)-(-5)\,dA = 0$ / check by integrating along y axis

24. Find the value of the line integral $\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y)=3yz\hat{\mathbf{i}}+3xz\hat{\mathbf{j}}+3xy\hat{\mathbf{k}}$.

(i) $\vec{\mathbf{r}}_1(t)=t\hat{\mathbf{i}}+9\hat{\mathbf{j}}+t\hat{\mathbf{k}},\quad 0\le t\le 81$ $\qquad \vec{F}=\langle 27t, 3t^2, 27t\rangle,\ d\vec{r}=\langle 1,0,1\rangle dt$

$$\int_0^{81} 3t\langle 9,t,9\rangle\cdot\langle 1,0,1\rangle dt = 54\int_0^{81} t\,dt = \frac{54\cdot 81^2}{2} = 27\cdot 81^2 = 3^{11}$$

(ii) $\vec{\mathbf{r}}_2(t)=t^2\hat{\mathbf{i}}+9\hat{\mathbf{j}}+t^2\hat{\mathbf{k}},\quad 0\le t\le 9$ $\qquad \vec{F}=\langle 27t^2, 3t^4, 27t^2\rangle,\ d\vec{r}=\langle 2t,0,2t\rangle dt$

$$\int_0^9 3t^2\langle 9,t^2,9\rangle\cdot\langle 2t,0,2t\rangle dt = 108\int_0^9 t^3\,dt = \frac{108}{4}9^4 = 27\cdot 9^4 = 3^{11}$$

25. Evaluate the line integral using the Fundamental Theorem of Line Integrals. Use a computer algebra system to verify your results.

$\int_C (4y\hat{\mathbf{i}}+4x\hat{\mathbf{j}})\cdot d\vec{r}$ $\quad f(x,y)=4xy$ potential function

$\int_C \vec{F}\cdot dr = f(9,4)-f(0,0)=144$

$C$: a smooth curve from (0,0) to (9,4)

26. Evaluate the line integral using the Fundamental Theorem of Line Integrals. Use a computer algebra system to verify your results.

$\int_C \frac{2x}{(x^2+y^2)^2}dx+\frac{2y}{(x^2+y^2)^2}dy$ $\quad f(x,y)=\frac{-1}{x^2+y^2}$ (potential function

$\int_C \vec{F}\cdot dr = f(-1,3)-f(13,3) = \frac{-1}{1+9} - \frac{-1}{169+9} = \frac{-42}{445}$

$C$: circle $(x-6)^2+(y-3)^2=49$ clockwise from (13,3) to (−1,3)

28. Use Green's Theorem to evaluate the integral

$\int_C (y-x)dx+(5x-y)dy = \iint_D 5-1\,dA = 4\int_0^{19} x-(x^2-18x)dx = 4\int_0^{19} 19x-x^2dx =$

for the path $C$: boundary of the region lying between the graphs of $y=x$ and $y=x^2-18x$

$4\left[\frac{19x^2}{2}-\frac{x^3}{3}\right]_0^{19}$

$\to 4\cdot 19^3\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{2}{3}19^3 = \frac{13718}{3}$

29. Use Green's Theorem to evaluate the integral

$\int_C 13xydx+(x+y)dy = \iint_D (1-13x)dA = \int_{-13}^{13}\int_0^{169-x^2}(1-13x)dydx = \int_{-13}^{13}(1-13x)(169-x^2)dx$

for the path $C$: boundary of the region lying between the graphs of $y=0$ and $y=169-x^2$.

$\to \int_{-13}^{13} +13x^3-x^2-13^3x+169dx$ drop the odd terms

$= \int_{-13}^{13} -x^2+169\,dx = \left[\frac{-x^3}{3}+169x\right]_{-13}^{13}$

$= 2\left(\frac{-13^3}{3}+13^3\right) = \frac{4}{3}13^3 = \frac{8788}{3}$

30. Use Green's Theorem to evaluate the integral

$$\int_C (x^2 - y^2)\,dx + 10xy\,dy = \iint_O 10y - (-2y)\,dA = \iint_O 12y\,dA = \int_0^{2\pi}\int_0^3 12r^2\sin\theta\,dr\,d\theta$$

for the path $C$: $x^2 + y^2 = 9$.

$$\Rightarrow [4r^3]_0^3 \, [-\cos\theta]_0^{2\pi} = 0$$

31. Use Green's Theorem to evaluate the integral

$$\int_C 8xy\,dx + 8(x+y)\,dy = 8\iint_{\odot} 1 - x\,dA = 8(\pi 4^2 - \pi 1^2) - \int_0^{2\pi}\int_1^4 r^2\cos\theta\,d\theta = 8\cdot 15\pi = 120\pi$$

$\rightarrow 0$ since $\int_0^{2\pi}\cos\theta\,d\theta = 0$

for $C$: boundary of the region lying between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$.

32. Use Green's Theorem to calculate the work done by the force $\vec{\mathbf{F}}$ on a particle that is moving counterclockwise around the closed path $C$.

$\vec{\mathbf{F}}(x, y) = 4xy\hat{\mathbf{i}} + (x+y)\hat{\mathbf{j}}$

$C: x^2 + y^2 = 9$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int 4xy\,dx + (x+y)\,dy = \iint_O 1 - 4x\,dA$$

$$= \pi 3^2 - 4\int_0^{2\pi}\int_0^3 r^2\cos\theta\,d\theta = 9\pi$$

since $\int_0^{2\pi}\cos\theta\,d\theta = 0$

33. Find the rectangular equation for the surface by eliminating parameters from the vector-valued function. Identify the surface.

$\vec{\mathbf{r}}(u, v) = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + \frac{v}{9}\hat{\mathbf{k}}$

$x = u$
$y = v$
$z = \frac{v}{9} = \frac{y}{9}$

$\Rightarrow 9z - y = 0$ a plane through $(0,0,0)$ with normal vector $\langle 0, -1, 9\rangle$