1. Find and simplify the function values.

$f(x,y) = 5 - x^2 - 10y^2$

(i) $f(0,0)$ (ii) $f(0,1)$ (iii) $f(3,9)$

(iv) $f(1,y)$ (v) $f(x,0)$ (vi) $f(t,1)$

(i) $f(0,0) = 5$

(ii) $f(0,1) = 5 - 10\cdot 1^2 = -5$

(iii) $f(3,9) = 5 - 3^2 - 10\cdot 9^2 = 5 - 9 - 810 = -814$

(iv) $f(1,y) = 5 - 1^2 - 10y^2 = 4 - 10y^2$

(v) $f(x,0) = 5 - x^2$

(vi) $f(t,1) = 5 - t^2 - 10\cdot 1^2 = -5 - t^2$

2. Find and simplify the function values.

$h(x,y,z) = \dfrac{xy}{z}$

(i) $h(7,8,24)$ (ii) $h(6,5,6)$ (iii) $h(-7,8,9)$ (iv) $h(10,9,-16)$

(i) $h(7,8,24) = \dfrac{7\cdot 8}{24} = \dfrac{7}{3}$

(ii) $h(6,5,6) = 6\cdot 5/6 = 5$

(iii) $h(-7,8,9) = -7\cdot 8/9 = -56/9$

(iv) $h(10,9,-16) = 10\cdot 9/-16 = -45/8$

3. Find and simplify the function values.

$g(x,y) = \int_x^y (16t-6)\,dt$

(i) $g(0,32)$ (ii) $g(1,32)$ (iii) $g\left(\frac{3}{8},32\right)$ (iv) $g\left(0,\frac{3}{8}\right)$

(i) $g(0,32) = \int_0^{32} (16t-6)\,dt = [8t^2 - 6t]_0^{32}$

$= 8\cdot 32^2 - 6\cdot 32 = 8000$

(ii) $g(1,32) = \int_1^{32} (16t-6)\,dt = [8t^2 - 6t]_1^{32} = 8000 - (8-6) = 7998$

(iii) $g(\frac{3}{8},32) = \int_{3/8}^{32} (16t-6)\,dt = [8t^2 - 6t]_{3/8}^{32} = 8000 - (8\cdot(\frac{3}{8})^2 - 6(\frac{3}{8})) = 8001.125$

(iv) $g(0,\frac{3}{8}) = \int_0^{3/8} (16t-6)\,dt = [8t^2 - 6t]_0^{3/8} = -9/8$

4. Describe the domain and range of the function.

$f(x,y) = \sqrt{100 - x^2 - y^2}$

Domain

$100 - x^2 - y^2 \geq 0$

$100 \geq x^2 + y^2$ disk centered at the origin with radius 10.

Range

$[0, 10]$

5. **Describe the level curves of the function. Sketch the level curves for the given c-values.**

$z = 6 - 2x - 3y,\ c = 0, 2, 4, 6$

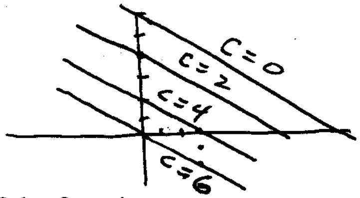
Level curves are straight lines

$L_0:\ 0 = 6 - 2x - 3y$

$L_2:\ 2 = 6 - 2x - 3y$

$L_4:\ 4 = 6 - 2x - 3y$

$L_6:\ 6 = 6 - 2x - 3y$



6. **Find the limit and discusss the continuity of the function.**

$$\lim_{(x,y)\to(-5,4)} (x + 7y^2) = (-5 + 7\cdot 4^2) = 107$$

polynomials are continuous througtout the entire domain.

7. **Find the limit and discusss the continuity of the function.**

$$\lim_{(x,y)\to(2,-10)} (8x + y + 2) = 8\cdot 2 + (-10) + 2 = 8$$

polynomials are continuous throughout the entire domain.

8. **Find the limit and discusss the continuity of the function.**

$$\lim_{(x,y)\to(1,1)} \frac{x}{\sqrt{7x+6y}}$$

The function is discontinuous on $7x + 6y \le 0$ ie $y \le -\frac{7}{6}x$

$$= \frac{1}{\sqrt{7\cdot 1 + 6\cdot 1}} = \frac{1}{\sqrt{13}}$$

9. **Find the limit and discusss the continuity of the function.**

$$\lim_{(x,y,z)\to(-1,0,8)} 5xe^{y^6 z}$$

continuous for all (x, y, z)

$$= 5\cdot(-1)\, e^{0^6\cdot 8} = -5$$

10. Find the limit (if it exists). If the limit does not exist, explain why.

$$\lim_{(x,y)\to(4,3)} \frac{xy-4}{3+xy}$$

discontinuous on $3+xy=0$ i.e. $y=\frac{-3}{x}$

$\Rightarrow$ continuous on open disk containing $(4,3)$

$\Rightarrow$ limit $= \frac{4\cdot 3-4}{3+4\cdot 3} = \frac{8}{15}$

11. Discusss the continuity of the function at the origin.

$$f(x,y)=\begin{cases} \frac{4x^8y^8}{x^8+y^8}, & (x,y)\neq(0,0) \\ 2, & (x,y)=(0,0) \end{cases}$$

discontinuous at the origin

because

$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0}{x^8+0} = 0 \neq 2 = f(0,0)$

12. Use polar coordinates to find the limit. [Hint: Let $x = r\cos\theta$ and $y = r\sin\theta$, and note that $(x,y)\to(0,0)$ implies $r\to 0$.]

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = \lim_{r\to 0} \frac{(r\cos\theta)^2(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2}$$

$$= \lim_{r\to 0} \frac{r^2\cos^2\theta\,\sin^2\theta}{\cos^2\theta+\sin^2\theta} = \lim_{r\to 0} r^2\cos^2\theta\sin^2\theta = 0$$

13. Discusss the continuity of the function.

$$f(x,y,z)=\frac{z}{x^2+y^2-64}$$

discontinuous on the cyclinder

$x^2+y^2-64=0$

i.e. $x^2+y^2=8^2$

14. Find each limit for the function $f(x,y)=-7x^2-4y$.

(i) $\lim_{\Delta x\to 0}\dfrac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$ $= \lim_{\Delta x\to 0} \dfrac{-7(x+\Delta x)^2-4y-(-7x^2-4y)}{\Delta x}$

$= \lim_{\Delta x\to 0} \dfrac{-14x\Delta x-7\Delta x^2}{\Delta x} = -14x$

(ii) $\lim_{\Delta y\to 0}\dfrac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$

$= \lim_{\Delta y\to 0} \dfrac{-7x^2-4(y+\Delta y)-(-7x^2-4y)}{\Delta y} = \lim_{\Delta y\to 0} \dfrac{-4\Delta y}{\Delta y} = -4$

15. Find both first partial derivatives.

$f(x,y)=-4x+3y-8$

$f_x = -4$

$f_y = 3$

16. Find both first partial derivatives.

$f(x,y)=-4x^2+5y^2+1$

$f_x = -8x$

$f_y = 10y$

17. Find both first partial derivatives.

$z=x\cdot\sqrt[4]{y} = xy^{\frac{1}{4}}$

$\dfrac{\partial z}{\partial x} = \sqrt[4]{y}$ $\qquad \dfrac{\partial z}{\partial y} = \dfrac{x}{4y^{3/4}} = \dfrac{x}{4\sqrt[4]{y^3}}$

18. Find both first partial derivatives.

$z=2y^3\sqrt{x} = 2y^3x^{\frac{1}{2}}$

$\dfrac{\partial z}{\partial x} = \dfrac{y^3}{\sqrt{x}}$ $\qquad \dfrac{\partial z}{\partial y} = 6y^2\sqrt{x}$

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19. **Find both first partial derivatives.**

$z = x^6 e^{10y}$

$$\frac{\partial z}{\partial x} = 6x^5 e^{10y} \qquad \frac{\partial z}{\partial y} = 10x^6 e^{10y}$$

20. **Find both first partial derivatives.**

$f(x, y) = \ln\left(x^5 + y^8\right)$

$$\frac{\partial f}{\partial x} = \frac{5x^4}{x^5+y^8} \qquad \frac{\partial z}{\partial y} = \frac{8y^7}{x^5+y^8}$$

21. **Find both first partial derivatives.**

$f(x, y) = \ln \sqrt[7]{xy}$ $= \dfrac{\ln(xy)}{7} = \dfrac{\ln x}{7} + \dfrac{\ln y}{7}$

$$\frac{\partial f}{\partial x} = \frac{1}{7x} \qquad \frac{\partial f}{\partial y} = \frac{1}{7y}$$

22. **Find both first partial derivatives.**

$z = \dfrac{x^2}{9y} + \dfrac{5y^2}{x}$

$$\frac{\partial z}{\partial x} = \frac{2x}{9y} - \frac{5y^2}{x^2}$$

$$\frac{\partial z}{\partial y} = \frac{10y}{x} - \frac{x^2}{9y^2}$$

23. **Find both first partial derivatives.**

$f(x, y) = \sqrt{4x^{10} + y^4}$

$$\frac{\partial f}{\partial x} = \frac{40x^9}{2\sqrt{4x^{10}+y^4}} = \frac{20x^9}{\sqrt{4x^{10}+y^4}}$$

$$\frac{\partial f}{\partial y} = \frac{2y^3}{\sqrt{4x^{10}+y^4}}$$

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24. Evaluate f_x and f_y at the given point.

$$f(x,y)=\frac{3xy}{\sqrt{9x^2+2y^2}},\quad (1,1)$$

$$\frac{\partial f}{\partial x}=\frac{3y\sqrt{9x^2+2y^2}-\frac{3xy\,18x}{2\sqrt{9x^2+2y^2}}}{\left(\sqrt{9x^2+2y^2}\right)^2}\Bigg|_{(1,1)}=\frac{3\sqrt{11}-\frac{3\cdot 18}{2\cdot\sqrt{11}}}{11}=\frac{6}{11\sqrt{11}}$$

$$\frac{\partial f}{\partial y}=\frac{3x\sqrt{9x^2+2y^2}-\frac{3xy\,4y}{2\sqrt{9x^2+2y^2}}}{9x^2+2y^2}=\frac{3\sqrt{11}-\frac{6}{\sqrt{11}}}{11}=\frac{27}{11\sqrt{11}}$$

25. For $f(x,y)$, find all values of x and y such that $f_x(x,y)=0$ and $f_y(x,y)=0$ simultaneously.

$$f(x,y)=9x^3-3xy+9y^3$$

$$f_x=27x^2-3y=0\Rightarrow y=9x^2$$

$$f_y=-3x+27y^2=0$$

$$\Rightarrow -3x+27(9x^2)^2=0\Rightarrow x=0 \text{ or } 27\cdot 81x^3=3\Rightarrow x=0 \text{ or } x=\frac{1}{9}$$

so $x=0,\ y=0$ or $x=\frac{1}{9},\ y=\frac{1}{9}$

26. Find the first partial derivatives with respect to x, y, and z.

$$w=\frac{2xz}{9x+6y}$$

$$\frac{\partial w}{\partial x}=\frac{2z(9x+6y)-18xz}{(9x+6y)^2}=\frac{14yz}{3(3x+2y)^2}$$

$$\frac{\partial w}{\partial y}=\frac{-2xz\cdot 6}{(9x+6y)^2}=\frac{-4xz}{3(3x+2y)^2}\qquad \frac{\partial w}{\partial z}=\frac{2x}{9x+6y}$$

27. Find the first partial derivatives with respect to x, y, and z.

$$H(x,y,z)=\cos(2x+8y+7z)$$

$$\frac{\partial H}{\partial x}=-2\sin(2x+8y+7z)$$

$$\frac{\partial H}{\partial y}=-8\sin(2x+8y+7z)$$

$$\frac{\partial H}{\partial z}=-7\sin(2x+8y+7z)$$

28. Evaluate f_x and f_y at the given point.

$$f(x,y,z)=\frac{xy}{x+y+z},\quad (6,8,3)$$

$$\frac{\partial f}{\partial x}=\frac{y(x+y+z)-xy}{(x+y+z)^2}=\frac{y(y+z)}{(x+y+z)^2}\Bigg|_{(6,8,3)}=\frac{88}{17^2}$$

$$\frac{\partial f}{\partial y}=\frac{x(x+y+z)-xy}{(x+y+z)^2}=\frac{x(x+z)}{(x+y+z)^2}\Bigg|_{(6,8,3)}=\frac{54}{17^2}$$

$$\frac{\partial f}{\partial z}=\frac{-xy}{(x+y+z)^2}\Bigg|_{(6,8,3)}=\frac{-48}{17^2}$$

29. **Find the four second partial derivatives. Observe that the second mixed partials are equal.**

$z = x^2 + 2xy + 8y^2$

$\frac{\partial z}{\partial x} = 2x + 2y \quad \frac{\partial z}{\partial y} = 2x + 16y$

$\frac{\partial^2 z}{\partial x^2} = 2, \ \frac{\partial^2 z}{\partial x \partial y} = 2, \ \frac{\partial^2 z}{\partial y^2} = 16, \ \frac{\partial^2 z}{\partial y \partial x} = 2$

30. **Find the four second partial derivatives. Observe that the second mixed partials are equal.**

$z = 11xe^y + 8ye^{-x}$

$\frac{\partial z}{\partial x} = 11e^y - 8ye^{-x}, \quad \frac{\partial z}{\partial y} = 11xe^y + 8e^{-x}$

$\frac{\partial^2 z}{\partial x^2} = 8ye^{-x}, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 11e^y - 8e^{-x}, \quad \frac{\partial^2 z}{\partial y^2} = 11xe^y$

31. **Find the total differential of the function $z = 5x^{10}y^9$.**

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 50x^9y^9dx + 45x^{10}y^8dy$

32. **Find the total differential of the function $z = \frac{x^9}{y}$.**

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{9x^8}{y}dx - \frac{x^9}{y^2}dz$

33. **Find the total differential of the function $z = -\frac{1}{x^7 + y^5}$.**

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{7x^6dx}{(x^7+y^5)^2} + \frac{5y^4dy}{(x^7+y^5)^2}$

34. **For the function $f(x,y) = 5x - 4y$:**

(i) Evaluate $f(1,5)$ and $f(1.09, 5.09)$ and calculate Δz, and

(ii) Use the total differential dz to approximate Δz.

$f(1,5) = 5 \cdot 1 - 4 \cdot 5 = -15$

$f(1.09, 5.09) = 5(1.09) - 4(5.09)$

$5.45 - 20.36 = -14.91$

$\Delta z = f(1.09, 5.09) - f(1,5)$

$= -14.91 - -15 = \boxed{.09} \doteq \Delta z$

$f_x(1,5) = 5, \ f_y(1,5) = -4$

$dz = 5dx - 4dy$

$= 5(.09) - 4(.09) = .45 - .36 = \boxed{.09} \doteq dz$

35. For the function $f(x,y) = x\sin y$:

$f(4,2) = 4\sin(2) = 3.637$

$f(4.08, 2.03) = 4.08\sin(2.03)$
$= 3.657$

(i) Evaluate $f(4,2)$ and $f(4.08,2.03)$ and calculate Δz, and

(ii) Use the total differential dz to approximate Δz.

$\Delta z = 3.657 - 3.637 = .020$

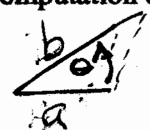
$$dz = f_x\,dx + f_y\,dy = (\sin 2)\cdot .08 + 4(\cos 2)\cdot .03$$
$$= .0228$$

36. The radius r and height h of a right circular cylinder are measured with possible errors of 8% and 3%, respectively. Approximate the maximum possible percent error in measuring the volume.

$$V = \pi r^2 h \qquad dV = 2\pi r h\,dr + \pi r^2\,dh = .16\pi r^2 h + .03\pi r^2 h$$
$$= .19\pi r^2 h$$

$$\frac{dV}{V} = .19 = 19\%$$

37. A triangle is measured and two adjacent sides are found to be 3 inches and 4 inches long, with an included angle of $\frac{\pi}{4}$. The possible errors in measurement are $\frac{1}{10}$ inch for the sides and 0.04 radian for the angle. Approximate the maximum possible error in the computation of the area.



$$\text{Area} = \frac{ab\sin\theta}{2} \qquad dA = \frac{b\sin\theta\,da}{2} + \frac{a\sin\theta\,db}{2} + \frac{ab\cos\theta\,d\theta}{2}$$

$$dA = \frac{4(\sin\frac{\pi}{4})}{2}\frac{1}{10} + \frac{3(\sin\frac{\pi}{4})}{2}\frac{1}{10} + \frac{3\cdot 4}{2}\left(\cos\frac{\pi}{4}\right)(.04) = \frac{1}{\sqrt{2}}\left(\frac{1}{5} + \frac{3}{20} + .24\right)$$
$$= \frac{.59}{\sqrt{2}} = .417$$

38. Let $w = xy$, where $x = 10\sin t$ and $y = -8\cos t$. Find $\frac{dw}{dt}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = y\,10\cos t + x\,8\sin t$$
$$= -80\cos^2 t + 80\sin^2 t = -80\cos(2t)$$

39. Let $w = \cos(2x - 4y)$, where $x = t^9$ and $y = 7$. Find $\frac{dw}{dt}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = -2\sin(2x-4y)\,9t^8 + 4\sin(2x-4y)\cdot 0$$
$$= -18t^8\sin(2t^9 - 28)$$

$w = t^9 t^2 t = t^{12} \quad \frac{dw}{dt} = 12t^{11}$

40. Let $w = xy\cos z$, where $x = t^9$, $y = t^2$, and $z = \arccos t$. Find $\frac{dw}{dt}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = (y\cos z)9t^8 + x(\cos z)2t + (xy\sin z)\frac{(-1)}{\sqrt{1-t^2}}$$

$$= 9t^{10}t + t^9 \cdot t \cdot 2t + t^9 t^2 \frac{\sin(\arccos t)}{\sqrt{1-t^2}} = 9t^{11} + 2t^{11} + t^{11} = 12t^{11}$$

41. Let $w = x^3 + y^3$, where $x = 4s + t$, $y = 4s - t$. Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ and evaluate each partial derivative at the point $s = 2$, $t = 2$. $x=10$, $y=6$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = 3x^2 \cdot 4 + 3y^2 4 = 12 \cdot 100 + 12 \cdot 36 = 1632$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = 3x^2 - 3y^2 = 3(100-36) = 192$$

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42. Let $w = x^7 - 7x^6 y$, where $x = e^s$, $y = e^t$. Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ and evaluate each partial derivative at the point $s = 0$, $t = 1$.

$$w = e^{7s} - 7e^{6s+t} \qquad \frac{\partial w}{\partial s} = 7e^{7s} - 7 \cdot 6e^{6s+t}\Big|_{\substack{s=0\\t=1}} = 7 - 42e$$

$$\frac{\partial w}{\partial t} = -7e^{6s+t}\Big|_{\substack{s=0\\t=1}} = -7e$$

43. Let $w = (x - y)^3$, where $x = r + \theta$ and $y = r - \theta$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$.

$$w = (r + \theta - (r - \theta))^3 = 8\theta^3 \qquad \frac{\partial w}{\partial r} = 0 \qquad \frac{\partial w}{\partial \theta} = 24\theta^2$$

44. Let $w = \frac{yz}{x}$, where $x = \theta^2$, $y = 9r + 7\theta$, and $z = 9r - 7\theta$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$.

$$w = \frac{(9r+7\theta)(9r-7\theta)}{\theta^2} = \frac{81r^2 - 49\theta^2}{\theta^2} = 81\frac{r^2}{\theta^2} - 49$$

$$\frac{\partial w}{\partial r} = 162r/\theta^2 \qquad \frac{\partial w}{\partial \theta} = -162\frac{r^2}{\theta^3}$$

45. Differentiate implicitly to find $\frac{dy}{dx}$.

$$F(x,y) = x^2 - 9xy + y^2 - 6x + y - 7 = 0$$

$$F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-2x + 9y + 6}{-9x + 2y + 1}$$

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46. Differentiate implicitly to find the first partial derivatives of z.

$F(x,y,z) = x^9 + y^9 + z^9 = 8$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-9x^8}{9z^8} = -\left(\frac{x}{z}\right)^8$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-9y^8}{9z^8} = -\left(\frac{y}{z}\right)^8$$

47. Differentiate implicitly to find the first partial derivatives of z.

$F(x,y,z) = x^3 + \sin(6y+7z) = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-3x^2}{7\cos(6y+7z)}, \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-6\cos(6y+7z)}{7\cos(6y+7z)} = -\frac{6}{7}$$

48. Differentiate implicitly to find the first partial derivatives of w.

$x^6 + y^6 + z^6 - 7yw + 9w^{10} = 10$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{-6x^5}{90w^9 - 7y}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{-6y^5 + 7w}{90w^9 - 7y}, \quad \frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = \frac{-6z^5}{90w^9 - 7y}$$

49. The radius of a right cylinder is increasing at a rate of 2 inches per minute, and the height is decreasing at a rate of 3 inches per minute. What is the rate of change of the volume and surface area (including both ends of the cylinder as well as the side) when the radius is 9 inches and height is 27 inches?

$V = \pi r^2 h \qquad \frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = 2\pi r h \cdot 2 + \pi r^2(-3) = 2\pi \cdot 9 \cdot 27 \cdot 2 + \pi \cdot 9^2(-3)$

$= 27^2\pi = 729\pi \text{ in}^3/\text{min}$

$S = 2\pi r h + 2\pi r^2 = 2\pi r(r+h) \qquad \frac{dS}{dt} = \frac{\partial S}{\partial r}\frac{dr}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt} = (2\pi h + 4\pi r)\cdot 2 + 2\pi r(-3) = (2\pi\cdot 27 + 4\pi\cdot 9)\cdot 2$

$+ 2\pi \cdot 9(-3)$

$= 126\pi$

50. Find the directional derivative of the function at P in the direction of $\vec{v}$.

$f(x,y) = 5x - 7xy + 10y, \quad P(1,4), \quad \vec{v} = \frac{1}{2}\left(\hat{i} + \sqrt{3}\hat{j}\right)$

$$D_v f(x,y) = \frac{\nabla f \cdot \vec{v}}{\|\vec{v}\|} = \frac{\langle 5-7y, -7x+10\rangle\big|_{(1,4)} \cdot \vec{v}}{1} = \langle -23, 3\rangle \cdot \left\langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right\rangle$$

$$= -\frac{23}{2} + \frac{3\sqrt{3}}{2} = -8.9$$

51. **Find the directional derivative of the function at P in the direction of $\vec{v}$.**

$f(x,y)=x^3-y^3,\quad P(2,1),\quad \vec{v}=\frac{\sqrt{2}}{2}(\hat{i}+\hat{j})$

$D_v f = \frac{\nabla f \cdot \vec{u}}{\|\vec{u}\|}$

$\nabla f = \langle 3x^2, -3y^2 \rangle \Big|_{(3,1)} = \langle 12, -3 \rangle$

$D_v f = \frac{\langle 12, -3 \rangle \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle}{1} = \frac{9}{\sqrt{2}}$

52. **Find the directional derivative of the function at P in the direction of $\vec{v}$.**

$f(x,y,z)=xy+yz+xz,\quad P(1,1,1),\quad \vec{v}=9\hat{i}+5\hat{j}-10\hat{k}$

$\nabla f = \langle y+z, x+z, y+x \rangle \Big|_{(1,1,1)} = \langle 2,2,2 \rangle$

$D_v f = \frac{\nabla f \cdot \vec{v}}{\|\vec{v}\|} = \frac{2\cdot 9 + 2\cdot 5 + 2\cdot 10}{\|\langle 9,5,-10 \rangle\|}$

$\frac{8}{\|\langle 9,5,-10 \rangle\|}$

$\frac{8}{\sqrt{81+25+100}} = \frac{8}{\sqrt{206}} = .557$

53. **Find the directional derivative of the function in the direction of $\vec{u}=\cos\theta\hat{i}+\sin\theta\hat{j}$.**

$f(x,y)=\sin(3x-2y),\quad \theta=\frac{\pi}{3} \Rightarrow u = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$\nabla f = \langle 3\cos(3x-2y), -2\cos(3x-2y) \rangle$

$D_u f = \nabla f \cdot \vec{u} = \left(\frac{3}{2} - \frac{1}{\sqrt{3}}\right)\cos(3x-2y) = .92265\cos(2x-2y)$

54. **Find the gradient of the function at the given point.**

$f(x,y)=9x-4y^2+2,\quad (4,1)$

$\nabla f = \langle 9, -8y \rangle \Big|_{(4,1)} = \langle 9, -8 \rangle$

55. **Find the gradient of the function at the given point.**

$g(x,y)=9xe^{\frac{y}{x}},\quad (9,0)$

$\nabla g = \langle 9e^{y/x} + 9x\left(-\frac{y}{x^2}\right)e^{y/x},\ 9x\frac{1}{x}e^{y/x} \rangle \Big|_{(9,0)}$

$= \langle 9, 9 \rangle$

56. Find the gradient of the function at the given point.

$w = 6x^2y - 10yz + z^2, \quad (1,1,-2)$

$$\nabla w = \langle 12xy,\ 6x^2 - 10z,\ -10y + 2z \rangle \Big|_{(1,1,-2)} = \langle 12,\ 6+20,\ -10-4 \rangle$$

$$= \langle 12,\ 26,\ -14 \rangle$$

57. Use the gradient to find the directional derivative of the function at P in the direction of Q.

$g(x,y) = x^2 + y^2 + 1, \quad P(2,4) \quad Q(6,12)$

$$\nabla g = \langle 2x, 2y \rangle \Big|_{(2,4)} = \langle 4, 8 \rangle$$

$$D_{PQ}\, g = \frac{\nabla g \cdot PQ}{\|PQ\|}$$

$$\frac{\langle 4,8 \rangle \cdot \langle 6-2, 12-4 \rangle}{\|\langle 6-2, 12-4 \rangle\|} = \frac{16+64}{\sqrt{16+64}}$$

$$= \sqrt{80} = 4\sqrt{5}$$

58. Use the gradient to find the directional derivative of the function at P in the direction of Q.

$f(x,y) = \sin(7x)\cos y, \quad P(0,0), \quad Q(\frac{\pi}{7}, \pi)$

$$\nabla f = \langle 7\cos(7x)\cos y,\ -\sin(7x)\sin y \rangle \Big|_{00} = \langle 7, 0 \rangle$$

$$D_{PQ}\, f = \frac{\nabla f \cdot PQ}{\|PQ\|} = \frac{\langle 7,0 \rangle \cdot \pi \langle \frac{1}{7}, 1 \rangle}{\pi \sqrt{\frac{1}{7}^2 + 1^2}} = \frac{1}{\sqrt{\frac{50}{49}}} = \frac{7}{5\sqrt{2}}$$

59. Find the gradient of the function and the maximum value of the directional derivative at the given point.

$w = xy^4z^7, \quad (9,1,1)$

$$\nabla w = \langle y^4z^7,\ 4xy^3z^7,\ 7xy^4z^6 \rangle \Big|_{(9,1,1)}$$

$$= \langle 1, 36, 63 \rangle$$

max directional derivative is $\|\nabla w\| = \sqrt{1 + 36^2 + 63^2}$

60. Find the directional derivative $D_u f(3,2)$ of the function $f(x,y) = 10 - \frac{x}{10} - \frac{y}{3}$ in the direction of $\vec{u} = \cos\theta\hat{i} + \sin\theta\hat{j}$.

$$\nabla f = \langle -\tfrac{1}{10}, -\tfrac{1}{3} \rangle$$

(i) $\theta = \frac{\pi}{4}$; (ii) $\theta = \frac{2\pi}{3}$

$$\vec{u} = \langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \rangle$$

$$\vec{u} = \langle -\tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle \quad D_u f = \nabla f \cdot \vec{u} = \frac{1}{20} - \frac{1}{2\sqrt{3}} = \frac{3 - 10\sqrt{3}}{60}$$

$$D_u f = \nabla f \cdot \vec{u} = -\frac{1}{10\sqrt{2}} - \frac{1}{3\sqrt{2}} = \frac{-13}{30\sqrt{2}}$$

61. Find the directional derivative $D_{\mathbf{u}} f(3,2)$ of the function $f(x,y) = 10 - \frac{x}{10} - \frac{y}{5}$ in the direction of $\vec{\mathbf{u}} = \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}$.

$\nabla f = \langle -\frac{1}{10}, -\frac{1}{5} \rangle$

(i) $\vec{\mathbf{v}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$; (ii) $\vec{\mathbf{v}} = -3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$

$D_v f = \frac{\nabla f \cdot \vec{v}}{\|\vec{v}\|} = \frac{\langle -\frac{1}{10}, -\frac{1}{5} \rangle \cdot \langle -3, -4 \rangle}{5} = \frac{11}{50}$

$D_v f = \frac{\nabla f \cdot \vec{v}}{\|\vec{v}\|} = \frac{\langle -\frac{1}{10}, -\frac{1}{5} \rangle \cdot \langle 1, 1 \rangle}{\| \langle 1, 1 \rangle \|} = \frac{-3}{10\sqrt{2}}$

62. Find the directional derivative $D_{\mathbf{u}} f(3,2)$ of the function $f(x,y) = 10 - \frac{x}{10} - \frac{y}{5}$ in the direction of $\vec{\mathbf{u}} = \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}$.

$\nabla f = \langle -\frac{1}{10}, -\frac{1}{5} \rangle$

(i) $\vec{\mathbf{v}}$ is the vector from (1,2) to $(-2,6)$; (ii) $\vec{\mathbf{v}}$ is the vector from (3,2) to (4,5).

$\vec{v} = \langle -3, 4 \rangle$

$D_v f = \frac{\nabla f \cdot v}{\|\vec{v}\|} = \frac{\langle -\frac{1}{10}, -\frac{1}{5} \rangle \cdot \langle -3, 4 \rangle}{5} = \frac{\frac{-5}{10}}{5} = \frac{-1}{10}$

$\vec{v} = \langle 1, 3 \rangle$

$D_v f = \frac{\nabla f \cdot \vec{v}}{\|\vec{v}\|} = \frac{\langle -\frac{1}{10}, -\frac{1}{5} \rangle \langle 1, 3 \rangle}{\sqrt{10}} = \frac{-7}{10\sqrt{10}}$

63. Find $\nabla f(x,y)$ for function $f(x,y) = 5 - \frac{x}{5} - \frac{y}{8}$.

$\nabla f = \langle -\frac{1}{5}, -\frac{1}{8} \rangle$

64. For function $f(x,y) = 6 - \frac{x}{6} - \frac{y}{9}$, find the maximum value of the directional derivative at (3,2).

$\max D_u f = \|\nabla f\| = \| \langle -\frac{1}{6}, -\frac{1}{9} \rangle \| = \sqrt{\frac{1}{36} + \frac{1}{81}}$

$= \sqrt{\frac{117}{36 \cdot 81}} = \sqrt{\frac{13}{36 \cdot 9}} = \frac{\sqrt{13}}{18}$

65. Use the gradient to find a normal vector to the graph of the equation at the given point.

$F(x,y) = 9x^2 - y = 4, \quad (10,896)$

$$\nabla F = \langle 18x, -1\rangle \Big|_{(10,896)} = \langle 180, -1\rangle$$

gradient is normal to the graph.

66. Find a unit normal vector to the surface $x+y+z=6$ at the point (3,0,3).

gradient is normal to surface

$\nabla f = \langle 1,1,1\rangle$ so unit normal is $\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle$

67. Find a unit normal vector to the surface $x^2+y^2+z^2=18$ at the point (4,1,1).

gradient is normal to the surface

$$\nabla F = \langle 2x, 2y, 2z\rangle \Big|_{(4,1,1)} = \langle 8,2,2\rangle$$

normalize: $\frac{\nabla F}{\|\nabla F\|} = \frac{\langle 8,2,2\rangle}{2\sqrt{18}} = \langle \frac{4}{\sqrt{18}}, \frac{1}{\sqrt{18}}, \frac{1}{\sqrt{18}}\rangle = \langle \frac{2\sqrt{2}}{3}, \frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}\rangle$

68. Find a unit normal vector to the surface $x^4y^2 - z = 0$ at the point $(1,6,36)$.

$$\nabla(x^4y^2 - z) = \langle 4x^3y^2, 2x^4y, -1\rangle \Big|_{(1,6,36)}$$

$= \langle 144, 12, -1\rangle$ unit vector is $\frac{1}{\sqrt{20881}}\langle 144, 12, -1\rangle$

69. Find a unit normal vector to the surface $x^2+2y+z^3=10$ at the point $(2,-1,2)$.

$$\nabla(x^2+2y+z^3) = \langle 2x, 2, 3z^2\rangle \Big|_{(2,-1,2)}$$

$= \langle 4, 2, 12\rangle$

normalizing we have $\frac{2\langle 2,1,6\rangle}{2\sqrt{4+1+36}} = \langle \frac{2}{\sqrt{41}}, \frac{1}{\sqrt{41}}, \frac{6}{\sqrt{41}}\rangle$

70. Find an equation of the tangent plane to the surface $g(x,y) = x^2 - y^2$ at the point (4,6,−20). normal vector is the gradient

$\nabla(g - z) = \langle 2x, -2y, -1\rangle \Big|_{(4,6,-20)} = \langle 8, -12, -1\rangle$ is normal to the surface

eqn. of tangent plane: $\langle 8,-12,-1\rangle \cdot (\langle x,y,z\rangle - \langle 4,6,-20\rangle) = 0$

$8(x-4) - 12(y-6) - (z+20) = 0$

71. Find an equation of the tangent plane to the surface $f(x,y) = 10 - \frac{5}{4}x - y$ at the point (8,−1,1). $\nabla(f - z) = \langle -\frac{5}{4}, -1, -1\rangle$ is a normal vector

tangent plane: $\langle -.8, -1, -1\rangle \cdot (\langle x,y,z\rangle - \langle 8,-1,1\rangle) = 0$

$0.8(x-8) + (y+1) + (z-1) = 0$, or $0.8x + y + z = 6.4$

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72. Find an equation of the tangent plane to the surface $x^2+9y^2+z^2=504$ at the point $(6,-6,12)$. $\nabla(x^2+9y^2+z^2)=\langle 2x,18y,2z\rangle\big|_{(6,-6,12)}=\langle 12,-108,24\rangle$

so an equivalent normal vector is $\frac{1}{12}\nabla F=\langle 1,-9,2\rangle$

tangent plane: $\langle 1,-9,2\rangle\cdot(\langle x,y,z\rangle-\langle 6,-6,12\rangle)=0$

$x-6-9(y+6)+2(z-12)=0$ or $x-9y+2z=84$

73. Find an equation of the tangent plane to the surface $x=y(7z-5)$ at the point $(96,6,3)$.

$\nabla(x+5y-7yz)=\langle 1,5-7z,-7y\rangle\big|_{(96,6,3)}=\langle 1,-16,-42\rangle$

tan. plane: $\langle 1,-16,-42\rangle\cdot(\langle x,y,z\rangle-\langle 96,6,3\rangle)=0$

$x-96-16(y-6)-42(z-3)=0$ or $x-16y-42z=-126$

74. Find an equation of the tangent plane and find symmetric equations of the normal line to the surface $x^2+y^2+z^2=201$ at the point $(1,10,10)$.

$\nabla(x^2+y^2+z^2)=\langle 2x,2y,2z\rangle\big|_{(1,10,10)}=\langle 2,20,20\rangle$

tangent plane $\langle 2,20,20\rangle\cdot(\langle x,y,z\rangle-\langle 1,10,10\rangle)=0$

$2(x-1)+20(y-10)+20(z-10)=0$ or $x+10y+10z=201$

normal line

$\langle x,y,z\rangle=\langle 1,10,10\rangle+t\langle 1,10,10\rangle$

$x-1=\frac{y-10}{10}=\frac{z-10}{10}$

75. Find an equation of the tangent plane and find symmetric equations of the normal line to the surface $xy-4z=0$ at the point $(-4,-10,10)$.

$\nabla(xy-4z)=\langle y,x,-4\rangle\big|_{(-4,-10,10)}=\langle -10,-4,-4\rangle$

tangent plane: $\langle -10,-4,-4\rangle\cdot(\langle x,y,z\rangle-\langle -4,-10,10\rangle)=0$

$-10(x+4)-4(y+10)-4(z-10)=0$ or $5x+2y+2z=-20$

normal line

$\langle x,y,z\rangle=\langle -4,-10,10\rangle+t\langle -10,-4,-4\rangle$

$\frac{x+4}{10}=\frac{y+10}{4}=\frac{z-10}{4}$

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76. Find an equation of the tangent plane and find symmetric equations of the normal line to the surface $xyz=30$ at the point $(1,3,10)$.

$\nabla(xyz)=\langle yz,xz,xy\rangle\big|_{(1,3,10)}=\langle 30,10,3\rangle$

tangent plane: $\langle 30,10,3\rangle\cdot(\langle x,y,z\rangle-\langle 1,3,10\rangle)=0$

$30(x-1)+10(y-3)+3(z-10)=0$

$30x+10y+3z=90$

normal line

$\langle x,y,z\rangle=\langle 1,3,10\rangle+t\langle 30,10,3\rangle$

$\frac{x-1}{30}=\frac{y-3}{10}=\frac{z-10}{3}$

77. Find the angle of inclination θ of the tangent plane to the surface $x^2-y^2+z=0$ at the point $(1,3,8)$.

$\nabla(x^2-y^2+z)=\langle 2x,-2y,1\rangle\big|_{(1,3,8)}=\langle 2,-6,1\rangle=\vec{n}$

$\theta=$ angle between $\langle 0,0,1\rangle$ and $\vec{n}$. $\cos\theta=\frac{\langle 2,-6,1\rangle\cdot\langle 0,0,1\rangle}{\|\langle 2,-6,1\rangle\|\,\|\langle 0,0,1\rangle\|}$

$=\frac{1}{\sqrt{4+36+1}}=\frac{1}{\sqrt{41}}$

$\theta=\arccos\left(\frac{1}{\sqrt{41}}\right)\approx 1.414$ radians

$\approx 81^\circ$

78. Find the angle of inclination θ of the tangent plane to the surface $3x^2+4y^2-z=0$ at the point $(6,6,252)$.

$\nabla(3x^2+4y^2-z) = \langle 6x, 8y, -1\rangle \big|_{(6,6,252)} = \langle 36,48,-1\rangle$

$\cos\theta = \left|\dfrac{\langle 36,48,-1\rangle\cdot\langle 0,0,1\rangle}{\|\langle 36,48,-1\rangle\|\,\|\langle 0,0,1\rangle\|}\right| = \dfrac{-1}{\sqrt{36^2+48^2+1}} \Rightarrow \theta = 1.554$ radians $\approx 89^\circ$

79. Find the distance from the point (0,0,0) to the plane $9x+10y+z=5$.

Normal vector for this plane is $\langle 9,10,1\rangle$

choose any vector to the plane, say $\langle 0,0,5\rangle$

then $\text{proj}_{\langle 9,10,1\rangle}\langle 0,0,5\rangle = \|\langle 9,10,1\rangle\|\cdot$ dist to plane (see Thm 10.13, pg 757)

$\Rightarrow \text{dist} = \dfrac{5}{\sqrt{81+100+1}} = \dfrac{5}{\sqrt{182}}$

80. Find three positive numbers x, y, and z whose sum is 3 and product is a maximum.

$z = 3-x-y$

$f(x,y) = xy(3-x-y)$

$f_x = 3y-2xy-y^2 = y(3-2x-y) = 0$

$f_y = 3x-x^2-2xy = x(3-x-2y) = 0$

$x=0$ and $y=0$ give minimums so $\begin{matrix}3-2x-y=0\\3-x-2y=0\end{matrix} \Rightarrow \begin{matrix}2x+y=3\\x+2y=3\end{matrix} \Rightarrow \begin{matrix}x=1\\y=1\end{matrix} \Rightarrow z=1$

this is the max

81. Find three positive numbers x, y, and z whose sum is 24 and the sum of the squares is a maximum.

$z = 24-x-y$, $f(x,y) = x^2+y^2+(24-x-y)^2$

$f_x = 2x+2(24-x-y)=0$
$f_y = 2y-2(24-x-y)=0$ $\Rightarrow$ $\begin{matrix}4x+2y=48\\2x+4y=48\end{matrix} \Rightarrow \begin{matrix}x=8\\y=8\end{matrix} \Rightarrow z=8$

82. The sum of the length (denote by z) and the girth (perimeter of a cross section) of packages carried by a delivery service cannot exceed 36 inches. Find the dimensions of the rectangular package of largest volume that may be sent.

$z+2x+2y=36 \Rightarrow z = 36-2x-2y$ $\quad V = xy(36-2x-2y)$

$V_x = 36y-4xy-2y^2 = 2y(18-2x-y)=0$
$V_y = 36x-4xy-2x^2 = 2x(18-2y-x)=0$ $\Rightarrow$ $\begin{matrix}2x+y=18\\x+2y=18\end{matrix} \Rightarrow \begin{matrix}x=6\\y=6\end{matrix} \Rightarrow z=12$

83. OMIT

84. A company manufactures two types of sneakers: running shoes and basketball shoes. The total revenue from x_1 units of running shoes and y_1 units of basketball shoes is:

$$R = -3x_1^2 - 8x_2^2 - 2x_1x_2 + 40x_1 + 109x_2 ,$$

where x_1 and x_2 are in thousands of units. Find x_1 and x_2 so as to maximize the revenue.

$R_{x_1} = -6x_1 - 2x_2 + 40 = 0 \Rightarrow x_2 = 20 - 3x_1$

$R_{x_2} = -16x_2 - 2x_1 + 109 = 0$

$\Rightarrow -16(20-3x_1) - 2x_1 + 109 = 0$

$\Rightarrow x_1 = \frac{211}{46} \Rightarrow x_2 = 20 - 3\left(\frac{211}{46}\right) = \frac{920-633}{46} = \frac{287}{46}$

85. Find the least squares regression line for the points (1,0), (3,3), (10,6).

$f(a,b) = (a1+b-0)^2 + (3a+b-3)^2 + (10a+b-6)^2$

$f_a = 2(a+b) + 2(3a+b-3)3 + 2(10a+b-6)10 = 0$

$f_b = 2(a+b) + 2(3a+b-3) + 2(10a+b-6) = 0$

$\Rightarrow 110a + 14b = 69$

$14a + 3y = 9$

$\Rightarrow a = \frac{81}{134}$

$b = \frac{12}{67}$

$134y = 81x + 24$