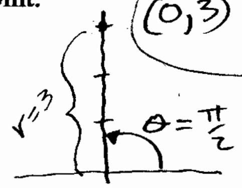
M252 pe1

Name: Key Date: ____________

1. For the given point in polar coordinates, find the corresponding rectangular coordinates for the point.

$\left(3, \frac{\pi}{2}\right)$

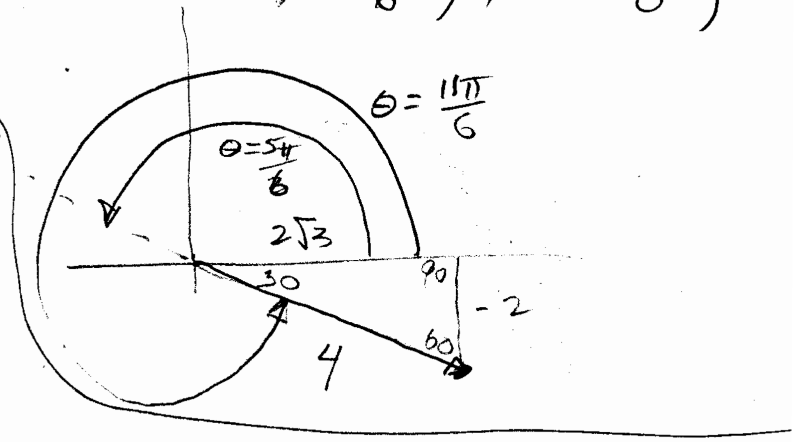


$(0,3) = (x,y)$

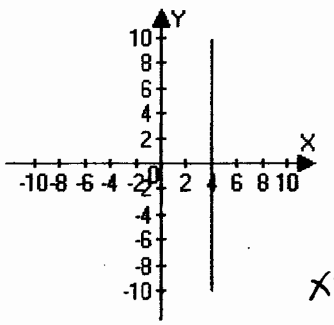
2. For the given point in rectangular coordinates, find two sets of polar coordinates for the point for $0 \le \theta \le 2\pi$.

$\left(2\sqrt{3}, -2\right)$

$\left(4, \frac{11\pi}{6}\right), \left(-4, \frac{5\pi}{6}\right)$



3. Match the graph with its polar equation.



$x = 4$

$r\cos\theta = 4$

$r = \frac{4}{\cos\theta}$

A) $r = 4\sin\theta$
B) $r = 8\cos(4\theta)$
C) $r = 5(1+\cos\theta)$
D) $r = 4\sec\theta$
E) $r = 5(1+\sin\theta)$

$r = 4\sec\theta$

4. Convert the rectangular equation to polar form.

$x = 3$

$r\cos\theta = 3$

$r = \frac{3}{\cos\theta}$

$r = 3\sec\theta$

5. Convert the rectangular equation to polar form.

$2x - y + 1 = 0$

$2r\cos\theta - r\sin\theta + 1 = 0$

now solve for r:

$r(2\cos\theta - \sin\theta) = -1$

$r = \frac{1}{\sin\theta - 2\cos\theta}$

6. **Convert the polar equation to rectangular form.**

$r=2 \Rightarrow r^2=4 \Rightarrow x^2+y^2=4$

check to make sure $r=-2$ is ok also.

7. **Convert the polar equation to rectangular form.**

$r=4\sin\theta$

$r^2=4r\sin\theta$

$x^2+y^2=4y$

$x^2+y^2-4y+4-4=0$

$x^2+(y-2)^2=4$

check to make sure of $r=0$.

Practice Problems 8 and 9

To find the intersection of two curves in polar coordinates:

r of curve one = r of curve 2 for the same θ.

8. Find the points of intersection of the graphs of the equations.

$$r = 1 + \cos\theta$$
$$r = 3\cos\theta$$

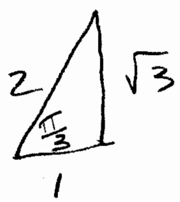
$1 + \cos\theta = 3\cos\theta$

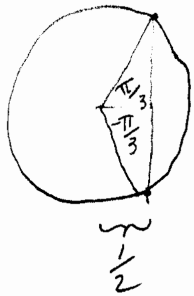
$1 = 2\cos\theta$

$\frac{1}{2} = \cos\theta$

$\theta = \frac{\pi}{3} + 2n\pi$

or $-\frac{\pi}{3} + 2n\pi$





$r = 1 + \cos\theta = 1 + \frac{1}{2} = \frac{3}{2}$

intersections at

$\left(\frac{3}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{3}{2}, -\frac{\pi}{3}\right)$

9. Find the points of intersection of the graphs of the equations.

$$r = \frac{\theta}{1.9}$$
$$r = 1.9$$

For first solution

$$\frac{\theta}{1.9} = 1.9 \Rightarrow \theta = 1.9^2 = 3.61$$

$(1.9, 3.61)$

The second curve is also given by

$r = -1.9$

so another solution is

$$\frac{\theta}{1.9} = -1.9 \Rightarrow \theta = -1.9^2 = -3.61$$

$(-1.9, -3.61)$

Practice Problems 10-12

Arc length in polar coordinates for $r = f(\theta)$

from pg 743 :

$$s = \int_{\alpha}^{\beta} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

10. Find the length of the curve over the given interval.

$r = 6\cos\theta, \; -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$f(\theta) = 6\cos\theta \qquad f'(\theta) = -6\sin\theta$

$$S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{36\cos^2\theta + 36\sin^2\theta}\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 6\, d\theta = \Big[6\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 6\pi$$

11. Find the length of the curve over the given interval.

$r = 7 + 7\sin\theta,\ 0 \le \theta \le 2\pi$

$$S = \int_0^{2\pi} \sqrt{(7+7\sin\theta)^2 + (7\cos\theta)^2}\, d\theta$$

$$= 7\sqrt{2} \int_0^{2\pi} \sqrt{1+\sin\theta}\, d\theta$$

remember cosine is sine of the complement

so $1 + \sin\theta = 1 + \cos\left(\frac{\pi}{2} - \theta\right)$

$= 2\cos^2\left(\frac{\frac{\pi}{2} - \theta}{2}\right)$ by $\frac{1}{2}$ angle formula

so

$$S = 7\sqrt{2} \int_0^{2\pi} \sqrt{2\cos^2\left(\frac{\frac{\pi}{2} - \theta}{2}\right)}\, d\theta$$

$$= 14 \int_0^{2\pi} \left|\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right| d\theta$$

$u = \frac{\pi}{4} - \frac{\theta}{2}$

$du = -\frac{d\theta}{2}$

$-2\,du = d\theta$

$$S = -28 \int_{\frac{\pi}{4}}^{-\frac{3\pi}{4}} |\cos u|\, du$$

$$S = -28 \int_{\frac{\pi}{4}}^{-\frac{\pi}{2}} \cos u \, du + 28 \int_{-\frac{\pi}{2}}^{-\frac{3\pi}{4}} \cos u \, du$$

$$= 28 \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos u \, du - \int_{-\frac{3\pi}{4}}^{-\frac{\pi}{2}} \cos u \, du \right]$$

$$= 28 \left[\left[\sin u \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \left[\sin u \right]_{-\frac{3\pi}{4}}^{-\frac{\pi}{2}} \right]$$

$$= 28 \left[\left(\frac{1}{\sqrt{2}} - -1 \right) - \left(-1 - -\frac{1}{\sqrt{2}} \right) \right]$$

$$= 28 \cdot 2 = 56$$

12. Find the length of the curve over the given interval.

$r = 3(1+\cos\theta),\ 0 \le \theta \le 2\pi$

$f(\theta) = 3(1+\cos\theta) \qquad f'(\theta) = -3\sin\theta$

$$s = \int_0^{2\pi} \sqrt{3^2(1+\cos\theta)^2 + 3^2\sin^2\theta}\, d\theta$$

$$= 3\sqrt{2} \int_0^{2\pi} \sqrt{1+\cos\theta}\, d\theta$$

From the double-angle formula

$\cos(2u) = 2\cos^2 u - 1$

we have

$1+\cos\theta = 2\cos^2\frac{\theta}{2}$

$$s = 6\int_0^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| d\theta$$

with $u = \frac{\theta}{2}$ $\qquad 2\,du = d\theta$

$$s = 12\int_0^{\pi} |\cos u|\, du = 12\left[\int_0^{\frac{\pi}{2}} \cos u\, du - \int_{\frac{\pi}{2}}^{\pi} \cos u\, du\right]$$

$$= 12\left(\left[\sin u\right]_0^{\frac{\pi}{2}} - \left[\sin u\right]_{\frac{\pi}{2}}^{\pi}\right)$$

$$= 12\left(1-0-(0-1)\right) = 24$$

13. Find vectors **u** and **v** whose initial and terminal points are given. Determine whether **u** and **v** are equivalent.

u: (3,8), (8,10) **v:** (5,3), (10,5)

$\vec{u} = \langle 5, 2 \rangle$ $\vec{v} = \langle 5, 2 \rangle$ $\therefore$ equivalent

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14. Find the vector **v** whose initial and terminal points are given below.

(6,4), (11,2) $\langle 11-6, 2-4\rangle = \langle 5, -2\rangle$

15. Find the vector **v** whose initial and terminal points are given below.

(3.7,7.7), (5.65,5.15) $\langle 5.65-3.7, 5.14-7.7\rangle = \langle 1.95, -2.56\rangle$

16. Find (a) 6**u** (b) **v**-**u** (c) 1**u**+4**v** given the following values for **u** and **v** .

$$\mathbf{u}=\langle 3,8\rangle,\quad \mathbf{v}=\langle 3,-2\rangle$$

(a) 6**u** $= \langle 6\cdot 3, 6\cdot 8\rangle$
$= \langle 18, 48\rangle$

(b) **v** - **u** $=$
$\langle 3-3, -2-8\rangle$
$= \langle 0, -10\rangle$

(c) 1**u**+4**v** $= \langle 3+4\cdot 3, 8+4(-2)\rangle$
$= \langle 15, 0\rangle$

17. The vector **v** and its initial point is given. Find the terminal point.

$\mathbf{v}=\langle -5,-4\rangle$, initial point (5,10)

$(5,10)+\langle -5,-4\rangle = (0,6)$

18. Find the magnitude of the vector given below.

$\mathbf{v}=\langle 3,5\rangle$ $\quad \|v\| = \sqrt{v\cdot v} = \sqrt{9+25} = \sqrt{34}$

19. Find the unit vector in the direction of **u**.

$\mathbf{u}=\langle 2,3\rangle$ $\quad \dfrac{u}{\|u\|} = \dfrac{\langle 2,3\rangle}{\sqrt{4+9}} = \left\langle \dfrac{2}{\sqrt{13}}, \dfrac{3}{\sqrt{13}}\right\rangle$

20. Given the vectors

$$\mathbf{u}=\langle 3,4\rangle, \quad \mathbf{v}=\langle -2,5\rangle$$

find the following:

(a) $\|\mathbf{u}+\mathbf{v}\|$

$\| \langle 3,4\rangle + \langle -2,5\rangle \|$

$= \| \langle 1,9\rangle \| = \sqrt{1+81} = \sqrt{82}$

(b) $\left\|\dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = \dfrac{\|u\|}{\|u\|} = 1$

(c) $\left\|\dfrac{\mathbf{u}+\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|}\right\| = \dfrac{\|u+v\|}{\|u+v\|} = 1$

21. Find the component form of a vector $\mathbf{v}$ given its magnitude and the angle it makes with the positive x-axis.

$\|\mathbf{v}\|=4, \quad \theta = 240^\circ$

$v = \langle 4\cos(240^\circ), 4\sin(240^\circ)\rangle$

$= \langle 4(-\frac{1}{2}), 4(-\frac{\sqrt{3}}{2})\rangle = \langle -2, -2\sqrt{3}\rangle$

22. Find the component form of a vector $\mathbf{v}$ given its magnitude of $\mathbf{u}$ and $\mathbf{u+v}$ and the angles that $\mathbf{u}$ and $\mathbf{u+v}$ make with the positive x-axis.

$\|\mathbf{u}\|=6, \quad \theta=30^\circ, \quad \|\mathbf{u}+\mathbf{v}\|=10, \quad \theta=240^\circ$

$u = \langle 6\cos(30^\circ), 6\sin(30^\circ)\rangle = \langle 6\frac{\sqrt{3}}{2}, 6\cdot\frac{1}{2}\rangle$

$u+v = \langle 10\cos(240^\circ), 10\sin(240^\circ)\rangle = \langle 10(-\frac{1}{2}), 10(\frac{-\sqrt{3}}{2})\rangle$

$v = (u+v) - u = \langle -5, -5\sqrt{3}\rangle - \langle 3\sqrt{3}, 3\rangle = \langle -5-3\sqrt{3}, -3-5\sqrt{3}\rangle$

23. Three forces with magnitudes 85 pounds, 90 pounds and 25 pounds act on an object at angles 80°, -40°, and 90°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant force. $85\langle\cos 80^\circ, \sin 80^\circ\rangle + 90\langle\cos -40^\circ, \sin -40^\circ\rangle + 25\langle\cos 90^\circ, \sin 90^\circ\rangle$

$=$

(The choices below are given to two decimal places.) $= \langle 83.70, 50.86\rangle$

24. Find the coordinates of the point that is located 5 units in front of the yz-plane, 7 units in front of the xz-plane, 3 units below the xy-plane. $(5, 7, -3)$

25. Find the distance between the points given below.

$(2, 3, 1), \quad (6, 5, 7)$

$\sqrt{(6-2)^2+(5-3)^2+(7-1)^2} = \sqrt{16+4+36} = \sqrt{56}$

26. Find the coordinates of the midpoint of the line segment joining the points given below.

$(-5, 4, -2)$, $(-3, 7, 2)$ $\quad \left(\frac{-5-3}{2}, \frac{4+7}{2}, \frac{-2+2}{2}\right) = (-4, 5.5, 0)$

27. Find the standard equation of the sphere with center $(4, 3, -4)$, and radius 4.

$(x-4)^2 + (y-3)^2 + (z+4)^2 = 16$

28. Find the standard equation of a sphere that has diameter with the end points given below.

center = midpoint = $\left(\frac{1+3}{2}, \frac{4+6}{2}, \frac{1+9}{2}\right) = (2,5,5)$

$(1, 4, 1)$, $(3, 6, 9)$ radius = distance $(3,6,9),(2,5,5) = \sqrt{(3-2)^2+(6-5)^2+(9-5)^2}$

$(x-2)^2 + (y-5)^2 + (z-5)^2 = 18$ $\quad = \sqrt{18}$

29. Complete the square to write the following equation in the standard equation of a sphere.

$x^2-6x+9-9+y^2-2y+1-1+z^2+8z+16-16+1=0$

$x^2 + y^2 + z^2 - 6x - 2y + 8z + 1 = 0$ $\quad (x-3)^2 + (y-1)^2 + (z+4)^2 = 25$

30. Find the component form of the vector **u** with the given initial and terminal points.

Initial point: $(4, 2, 5)$ $\quad \vec{u} = \langle 7-4, -2-2, 8-5 \rangle$

Terminal point: $(7, -2, 8)$ $\quad = \langle 3, -4, 3 \rangle$

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31. Given the vector **v** and its initial point find the terminal point of the vector.

$\mathbf{v} = \langle -3, -3, 1 \rangle$, initial point $(3, 5, -1)$

terminal point
$= (3-3, 5-3, -1+1)$
$= (0, 2, 0)$

32. Find the vector **z=4v+4u-5w** given that:

$\mathbf{v} = \langle 5, -1, 6 \rangle$, $\mathbf{u} = \langle 3, -1, 5 \rangle$, $\mathbf{w} = \langle -5, -5, 6 \rangle$

$z = \langle 4\cdot5, 4(-1), 4\cdot6 \rangle + \langle 4\cdot3, 4(-1), 4\cdot5 \rangle + \langle -5(-5), -5(-5), -5(6) \rangle$

$\to \langle 20+12+25, -4-4+25, 24+20-30 \rangle$

$= \langle 57, 17, 14 \rangle$

33. Find the magnitude of the vector given below.

$\mathbf{v} = \langle 0, -2, -3 \rangle$

$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{0^2+(-2)^2+(-3)^2} = \sqrt{4+9} = \sqrt{13}$

34. Find the magnitude of the vector **v** given its initial and terminal points.

Initial point: $(-3,-4,-6)$

Terminal point: $(-8,1,-9)$

$\|\langle -8--3,\ 1--4,\ -9--6\rangle\| = \|\langle -5,5,-3\rangle\|$

$= \sqrt{25+25+9} = \sqrt{59}$

35. Find the unit vector in the direction of **u**.

$\mathbf{u} = \langle -5,-3,4\rangle$

$\|u\| = \sqrt{25+9+16} = \sqrt{50} = 5\sqrt{2}$

$\frac{u}{\|u\|} = \left\langle \frac{-5}{5\sqrt{2}}, \frac{-3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}} \right\rangle$

36. Find (a) $\mathbf{u}\cdot\mathbf{v}$ (b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{v}$ (c) $\mathbf{u}\cdot(3\mathbf{v})$ given the vectors **u** and **v**.

$\mathbf{u} = \langle 2,7\rangle, \quad \mathbf{v} = \langle 5,5\rangle$

(a) $\mathbf{u}\cdot\mathbf{v}$

$2\cdot5+7\cdot5=45$

(b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{v}$

$45\langle 5,5\rangle = \langle 225,225\rangle$

(c) $\mathbf{u}\cdot(3\mathbf{v})$

$\langle 2,7\rangle\cdot\langle 15,15\rangle = 2\cdot15+7\cdot15$

$= 9\cdot15 = 135$

37. Find the angle between the vectors for **u** and **v** given below.

$\mathbf{u} = \langle 1,5\rangle, \quad \mathbf{v} = \langle -3,-1\rangle$

$u\cdot v = 1(-3)+5(-1) = -8$

$\|u\| = \sqrt{1+25} = \sqrt{26}$

$\|v\| = \sqrt{9+1} = \sqrt{10}$

$\cos\theta = \frac{u\cdot v}{\|u\|\ \|v\|}$

$= \frac{-8}{\sqrt{260}} = \frac{-4}{\sqrt{65}}$

$\theta = \arccos\left(\frac{-4}{\sqrt{65}}\right)$

38. Find the angle between the vectors for **u** and **v**.

$\mathbf{u} = -4\mathbf{i}+7\mathbf{j}, \quad \mathbf{v} = -5\mathbf{i}+5\mathbf{j}$

$u\cdot v = (-4)(-5)+7\cdot5 = 55$

$\|u\| = \sqrt{16+49} = \sqrt{65}$

$\|v\| = \sqrt{25+25} = \sqrt{50}$

$\cos\theta = \frac{u\cdot v}{\|u\|\ \|v\|}$

$= \frac{55}{\sqrt{65}\sqrt{50}} = \frac{11}{\sqrt{130}}$

$\theta = \arccos\left(\frac{11}{\sqrt{130}}\right)$

39. Determine whether **u** and **v** are orthogonal, parallel, or neither.

$\mathbf{u} = \langle 20,5\rangle, \quad \mathbf{v} = \langle 3,-12\rangle$

$u\cdot v = 20\cdot3+5(-12) = 0 \Rightarrow$ orthogonal

40. Determine whether **u** and **v** are orthogonal, parallel, or neither.

$\mathbf{u} = 12\mathbf{i}+4\mathbf{j}, \quad \mathbf{v} = 5\mathbf{i}-15\mathbf{j}$

$u\cdot v = 12\cdot5+4(-15) = 0 \Rightarrow$ orthogonal

41. Determine whether **u** and **v** are orthogonal, parallel, or neither.

$$\mathbf{u} = \langle 4,6\rangle, \quad \mathbf{v} = \langle -12,-18\rangle$$

$\|u\| = \sqrt{16+36} = \sqrt{52}$

$\|v\| = \sqrt{144+324} = \sqrt{468}$

$u \cdot v = 4(-12) + 6(-18) = -48 - 108 = -156$

$\cos\theta = \frac{u \cdot v}{\|u\| \|v\|}$

$= \frac{-156}{\sqrt{52 \cdot 468}} = -1$

$\Rightarrow$ parallel

42. Find the direction cosines of the vector **u** given below.

$$\mathbf{u} = \langle 6,4,-5\rangle$$

$\|u\| = \sqrt{36+16+25} = \sqrt{77}$

$\frac{6}{\sqrt{77}}, \frac{4}{\sqrt{77}}, \frac{-5}{\sqrt{77}}$

43. Find the direction cosines of the vector **u** given below.

$$\mathbf{u} = -4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

$\|u\| = \sqrt{16+4+9} = \sqrt{29}$

$\frac{-4}{\sqrt{29}}, \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}$

44. Find the direction angles of the vector **u** given below.

$$\mathbf{u} = \langle 6,2,3\rangle$$

$\|u\| = \sqrt{36+4+9} = \sqrt{49} = 7$

$\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$

45. Find the projection of **u** onto **v**, and the vector component of **u** orthogonal to **v**.

$$\mathbf{u} = \langle -7,9\rangle, \quad \mathbf{v} = \langle 7,5\rangle$$

$u \cdot v = -49 + 45 = -4$

$u \cdot u = 49 + 81 = 130$

$v \cdot v = 49 + 25 = 74$

Projection of u onto v

$\frac{u \cdot v}{v \cdot v} v = \frac{-4}{74}\langle 7,5\rangle = \left\langle \frac{-14}{37}, \frac{-10}{37} \right\rangle$

Component of u orthogonal to v

$u - \left\langle \frac{-14}{37}, \frac{-10}{37} \right\rangle = \left\langle \frac{-245}{37}, \frac{343}{37} \right\rangle$

46. Find the projection of **u** onto **v**, and the vector component of **u** orthogonal to **v**.

$$\mathbf{u} = \langle -1,7,5\rangle, \quad \mathbf{v} = \langle 5,10,2\rangle$$

$u \cdot v = -5 + 70 + 10 = 75$

$u \cdot u = 1 + 49 + 25 = 75$

$v \cdot v = 25 + 100 + 4 = 129$

Projection of u onto v

$\frac{u \cdot v}{v \cdot v} v = \frac{75}{129}\langle 5,10,2\rangle$

$\left\langle \frac{125}{43}, \frac{250}{43}, \frac{50}{43} \right\rangle$

Component of u orthogonal to v

$u - \frac{u \cdot v}{v \cdot v} v$

$= \left\langle -\frac{168}{43}, \frac{51}{43}, \frac{165}{43} \right\rangle$