

1. Find the gradient vector for the scalar function. (That is, find the conservative vector field for the potential function.)

$$f(x,y) = 7x^2 + 8xy + 2y^2 \quad \nabla f = \langle 14x + 8y, 8x + 4y \rangle \\ \text{or } (14x + 8y)\hat{i} + (8x + 4y)\hat{j}$$

2. Find the gradient vector for the scalar function. (That is, find the conservative vector field for the potential function.)

$$f(x,y) = \sin 9x \cos 3y \quad \nabla f = \langle 9\cos 9x \cos 3y, -3\sin 9x \sin 3y \rangle$$

3. Determine whether the vector field is conservative.

$$\vec{F}(x,y) = 9y^2(7y\hat{i} - x\hat{j}) \quad N = -9xy^2 \quad M = 63y^3$$

$$\frac{\partial N}{\partial x} = -9y^2 \quad \frac{\partial M}{\partial y} = 189y^2$$

A) Conservative
B) Not Conservative

$$\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$$

4. Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

$$\vec{F}(x,y) = 7x^6y\hat{i} + x^7\hat{j} \quad \frac{\partial}{\partial y} 7x^6y = 7x^6 \quad \frac{\partial}{\partial x} x^7 = 7x^6 \Rightarrow \text{conservative}$$

$$f(x,y) = x^7y + C$$

5. Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

$$\vec{F}(x,y) = \frac{6y}{x}\hat{i} - \frac{x^6}{y^6}\hat{j} \quad \frac{\partial}{\partial y} \frac{6y}{x} = \frac{6}{x} \quad \frac{\partial}{\partial x} -\frac{x^6}{y^6} = -\frac{6x^5}{y^6} \Rightarrow \text{NOT conservative}$$

6. Find the curl for the vector field at the given point.

$$\vec{F}(x,y,z) = 2xyz\hat{i} + 2y\hat{j} + 2z\hat{k}, \quad (2,3,2)$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & 2y & 2z \end{vmatrix} = 0\hat{i} + 2xz\hat{j} - 2xz\hat{k}$$

at $(2,3,2)$ we have

$$\text{curl } \vec{F} = \langle 0, 12, -8 \rangle$$

7. Determine whether the vector field is conservative. If it is, find a potential function for the vector field.
- $$\frac{\partial M}{\partial y} = 12x^2y^3z^5 = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = 15x^2y^4z^4 = \frac{\partial P}{\partial x},$$
- $$\vec{F}(x, y, z) = 3x^2y^4z^5\hat{i} + 4x^3y^3z^5\hat{j} + 5x^3y^4z^4\hat{k}$$
- $$\frac{\partial N}{\partial z} = 20x^3y^3z^4 = \frac{\partial P}{\partial y}$$
- $$f(x, y, z) = x^3y^4z^5$$

8. Find the divergence of the vector field.

$$\vec{F}(x, y, z) = 9x^4\hat{i} - xy^3\hat{j}$$

$$\frac{\partial 9x^4}{\partial x} + \frac{\partial -xy^3}{\partial y} = 36x^3 - 3xy^2$$

9. Find the divergence of the vector field at the given point.

$$\vec{F}(x, y, z) = 8xyz\hat{i} + 8yz\hat{j} + 8x\hat{k}, \quad (8, 9, 8)$$

$$\frac{\partial 8xyz}{\partial x} + \frac{\partial 8yz}{\partial y} + \frac{\partial 8x}{\partial z} = 8yz + 8$$

at (8, 9, 8), $\nabla \cdot F = 8 \cdot 9 \cdot 8 + 8 = 584$

10. Find $\text{curl}(\vec{F} \times \vec{G})$.

$$\vec{F}(x, y, z) = 7\hat{i} + 8x\hat{j} + 9y\hat{k}$$

$$\vec{G}(x, y, z) = 7x\hat{i} - 7y\hat{j} + 7z\hat{k}$$

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11. Find $\text{div}(\vec{F} \times \vec{G})$.

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$$\vec{F}(x, y, z) = 6\hat{i} + 7x\hat{j} + 8y\hat{k}$$

$$\vec{G}(x, y, z) = 6x\hat{i} - 6y\hat{j} + 6z\hat{k}$$

12. Evaluate the line integral along the given path.

$$\int_C (2x - 7y) ds \quad C: \vec{r}(t) = 7\hat{i} + 3t\hat{j}, \quad 0 \leq t \leq 8$$

$$\vec{r}'(t) = \langle 0, 3 \rangle \quad \| \vec{r}'(t) \| = \sqrt{49+9} = \sqrt{58} \quad ds = \| \vec{r}'(t) \| dt$$

$$= \sqrt{58} dt$$

$$\int_C (2x - 7y) ds = \int_0^8 (2(7t) - 7(3t)) \sqrt{58} dt$$

$$\sqrt{58} \int_0^8 -7t dt = \sqrt{58} \left(-7 \cdot \frac{8^2}{2} \right) = -224\sqrt{58}$$

13. Evaluate $\int_C (x^2 + y^2) ds$ where the path C is:

(i) the x-axis from $x = 0$ to $x = 1$; $x = t \quad y = 0 \quad r(t) = \langle t, 0 \rangle \quad r'(t) = \langle 1, 0 \rangle$

$$ds = \|r'(t)\| dt = dt \quad \int_0^1 t^2 + 0^2 dt = \left[\frac{t^3}{3} \right]_0^1 = \frac{1}{3}$$

(ii) the y-axis from $y = 1$ to $y = 3$. $x = 0, y = t, 1 \leq t \leq 3, \|r'(t)\| = 1 \quad ds = dt$

$$\int_1^3 0^2 + t^2 dt = \left[\frac{t^3}{3} \right]_1^3 = 9 - \frac{1}{3} = \frac{26}{3}$$

14. Evaluate $\int_C (x + 49\sqrt{y}) ds$ where the path C is:

(i) the line from $(0,0)$ to $(1,1)$; $r(t) = \langle t^2, t^2 \rangle, r'(t) = \langle 2t, 2t \rangle, ds = 2\sqrt{2}t dt$

$$\int_0^1 (t^2 + 49t) 2\sqrt{2}t dt = 2\sqrt{2} \int_0^1 t^3 + 49t^2 dt = 2\sqrt{2} \left[\frac{t^4}{4} + \frac{49t^3}{3} \right]_0^1 = 2\sqrt{2} \left(\frac{1}{4} + \frac{49}{3} \right) = 199\sqrt{2}/6$$

(ii) the line from $(0,0)$ to $(3,9)$. $r(t) = \langle 3t^2, 9t^2 \rangle, 0 \leq t \leq 1, r'(t) = \langle 6t, 18t \rangle$

$$\int_0^1 (3t^2 + 3 \cdot 49t) 6t \sqrt{10} dt = 18\sqrt{10} \int_0^1 t^3 + 49t^2 dt = 18\sqrt{10} \left[\frac{t^4}{4} + \frac{49t^3}{3} \right]_0^1 = 18\sqrt{10} \left(\frac{1}{4} + \frac{49}{3} \right) = 597\sqrt{10}/2$$

15. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where C is represented by $\hat{r}(t)$.

$$\bar{F}(x, y) = xy\hat{i} + y\hat{j}$$

$$C: \hat{r}(t) = 32\hat{i} + \hat{j}, 0 \leq t \leq 8$$

$$d\hat{r} = \langle 32, 1 \rangle dt$$

$$\int_0^8 \langle xy, y \rangle \cdot \langle 32, 1 \rangle dt \rightarrow \int_0^8 (32^2 t^2 + t) dt = \frac{32^2 \cdot 8^3}{3} + \frac{8^2}{2} = 174794 \frac{2}{3}$$

16. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where C is represented by $\hat{r}(t)$.

$$\bar{F}(x, y) = 5x\hat{i} + 9y\hat{j}$$

$$C: \hat{r}(t) = \hat{i} + \sqrt{4-t^2}\hat{j}, -2 \leq t \leq 2$$

$$d\hat{r} = \left\langle 1, \frac{-t}{\sqrt{4-t^2}} \right\rangle dt$$

$$\int_{-2}^2 \langle 5t, 9\sqrt{4-t^2} \rangle \cdot \left\langle 1, \frac{-t}{\sqrt{4-t^2}} \right\rangle dt = \int_{-2}^2 (5t - 9t) dt = \left[-2t^2 \right]_{-2}^2 = 0$$

17. Evaluate the line integral along the path C given by $x = 4t, y = 20t$, where $0 \leq t \leq 1$.

$$\int_C (x+4y^2) ds \quad \hat{r}(t) = 4t\langle 1, 5 \rangle \quad 0 \leq t \leq 1 \quad r(t) = 4\langle 1, 5 \rangle$$

$$ds = 4\sqrt{26} dt$$

$$\int_0^1 (4t + 4(20t)^2) 4\sqrt{26} dt = 16\sqrt{26} \int_0^1 t + 400t^2 dt = 16\sqrt{26} \left[\frac{t^2}{2} + \frac{400t^3}{3} \right]_0^1 = 16\sqrt{26} \left(\frac{1}{2} + \frac{400}{3} \right) = \frac{6424}{3}\sqrt{26}$$

18. Evaluate the line integral

$$\int_C (2x - y) dx + (x + 3y) dy$$

along the path C , where C is:

(i) the x -axis from $x = 0$ to $x = 8$. $r(t) = \langle t, 0 \rangle$, $dx = dt$, $dy = 0$

$$\int_0^8 (2t - 0) dt + (t + 3 \cdot 0) \cdot 0 = \left[t^2 \right]_0^8 = 64$$

(ii) the y -axis from $y = 0$ to $y = 12$. $r(t) = \langle 0, t \rangle$, $dx = 0$, $dy = dt$

$$\int_0^{12} (2 \cdot 0 - t) \cdot 0 + (0 + 3t) dt = \left[\frac{3t^2}{2} \right]_0^{12} = 216$$

19. Find the area of the lateral surface over the curve C in the xy -plane and under the surface $z = f(x, y)$, where

$$r(t) = \langle 5t, 6t \rangle \quad ds = \sqrt{25+36} dt$$

$$\text{Lateral surface area} = \int_C f(x, y) ds$$

$$f(x, y) = 4, C: \text{line from } (0, 0) \text{ to } (5, 6).$$

$$\int_0^1 4\sqrt{61} dt = 4\sqrt{61}$$

20. Set up and evaluate the integral $\int_C \vec{F} \cdot d\vec{r}$ for each parametric representation of C .

$$\vec{F}(x, y) = x^2 \hat{i} + xy \hat{j} \quad \text{on } C \quad \vec{F} = \langle 9t^2, 21t^3 \rangle$$

$$(i) \vec{r}_1(t) = 3\hat{i} + 7t^2\hat{j}, \quad 0 \leq t \leq 1 \quad d\vec{r} = \langle 3, 14t \rangle dt$$

$$(ii) \vec{r}_2(t) = 3\sin\theta\hat{i} + 7\sin^2\theta\hat{j}, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$\text{on } C \quad \vec{F} = \langle 9\sin^2\theta, 21\sin^3\theta \rangle$$

$$d\vec{r} = \langle 3\cos\theta, 14\sin\theta\cos\theta \rangle d\theta$$

$$\begin{aligned} & \int_0^1 3t^2 \langle 3, 7t \rangle \cdot \langle 3, 14t \rangle dt \\ &= 3 \int_0^1 9t^2 + 98t^4 dt \\ &= 3 \left[3t^3 + \frac{98}{5}t^5 \right]_0^1 \\ &= 3 \left(3 + \frac{98}{5} \right) = 67.8 \end{aligned}$$

$$\int_0^{\frac{\pi}{2}} 3\sin^2\theta\cos\theta \langle 3, 7\sin\theta \rangle \cdot \langle 3, 14\sin\theta \rangle d\theta$$

$$\begin{aligned} &= 3 \int_0^{\frac{\pi}{2}} \sin^2\theta\cos\theta (9 + 98\sin^2\theta) d\theta \quad u = \sin\theta \\ &\quad du = \cos\theta d\theta \\ &= 3 \int_0^1 u^2 (9 + 98u^2) du = 3 \left[3u^3 + \frac{98}{5}u^5 \right]_0^1 = 3 \left(3 + \frac{98}{5} \right) \\ &= 67.8 \end{aligned}$$

21. Determine whether or not the vector field is conservative.

$$\vec{F}(x, y) = 40x^7y^7\hat{i} + 35x^8y^6\hat{j}$$

- A) Conservative
B) Not conservative

$$\frac{\partial M}{\partial y} = 280x^7y^6 \quad \frac{\partial N}{\partial x} = 280x^7y^6$$

22. Find the value of the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = 2xy\hat{i} + x^2\hat{j}$.

(i) $\vec{r}_1(t) = t\hat{i} + t^3\hat{j}, \quad 0 \leq t \leq 1 \quad \vec{F} = \langle 2t^4, t^2 \rangle, \quad d\vec{r} = \langle 1, 3t^2 \rangle dt$
 $\int_0^1 t^2 \langle 2t^4, 1 \rangle \cdot \langle 1, 3t^2 \rangle dt = \int_0^1 5t^4 dt = [t^5]_0^1 = 1$

(ii) $\vec{r}_2(t) = t\hat{i} + t^7\hat{j}, \quad 0 \leq t \leq 1 \quad \vec{F} = \langle 2t^8, t^2 \rangle, \quad d\vec{r} = \langle 1, 7t^6 \rangle dt$
 $\int_0^1 t^2 \langle 2t^8, 1 \rangle \cdot \langle 1, 7t^6 \rangle dt = \int_0^1 9t^8 dt = [t^9]_0^1 = 1$

23. Find the value of the line integral $\int_C (2x - 5y + 7)dx - (5x + 7y - 8)dy$.

(i) C: the curve $x = \sqrt{49 - y^2}$ from $(0, -7)$ to $(0, 7)$ $x = 7\cos\theta, y = 7\sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
 $d\vec{r} = \langle -7\sin\theta, 7\cos\theta \rangle$
 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2(7\cos\theta) - 5(7\sin\theta) + 7)(-7\sin\theta) - (5(7\cos\theta) + 7(7\sin\theta) - 8)(7\cos\theta) d\theta$ factor out 7 and drop odd terms
 $= 7 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (35(\sin^2\theta - \cos^2\theta) + 8\cos\theta) d\theta = 7 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -35\cos(2\theta) + 8\cos\theta d\theta = 7 \left[\frac{-35}{2}\sin(2\theta) + 8\sin\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 112$

(ii) C: from $(0, -7)$ to $(0, 7)$ along $x = \sqrt{49 - y^2}$, then back to $(0, -7)$ along the y-axis (a closed curve).

Using Green's Theorem $\int_C (-5) - (-5) dA = 0$ / check by integrating along y-axis

24. Find the value of the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = 3yz\hat{i} + 3xz\hat{j} + 3xy\hat{k}$.

(i) $\vec{r}_1(t) = t\hat{i} + 9\hat{j} + t\hat{k}, \quad 0 \leq t \leq 81 \quad \vec{F} = \langle 27t, 3t^2, 27t \rangle, \quad d\vec{r} = \langle 1, 0, 1 \rangle dt$
 $\int_0^{81} 3t \langle 9, t, 9 \rangle \cdot \langle 1, 0, 1 \rangle dt = 54 \int_0^{81} t dt = \frac{54 \cdot 81^2}{2} = 27 \cdot 81^2 = 3''$

(ii) $\vec{r}_2(t) = t^2\hat{i} + 9\hat{j} + t^2\hat{k}, \quad 0 \leq t \leq 9 \quad \vec{F} = \langle 27t^2, 3t^4, 27t^2 \rangle, \quad d\vec{r} = \langle 2t, 0, 2t \rangle dt$

$$\int_0^9 3t^2 \langle 9, t^2, 9 \rangle \cdot \langle 2t, 0, 2t \rangle dt = 108 \int_0^9 t^3 dt = \frac{108}{4} 9^4 = 27 \cdot 9^4 = 3''$$

25. Evaluate the line integral using the Fundamental Theorem of Line Integrals. Use a computer algebra system to verify your results.

$$\int_C (4y\hat{i} + 4x\hat{j}) \cdot d\vec{r} \quad f(x,y) = 4xy \quad \text{potential function}$$

$$\int_C \vec{F} \cdot d\vec{r} = f(9,4) - f(0,0) = 144$$

C: a smooth curve from (0,0) to (9,4)

26. Evaluate the line integral using the Fundamental Theorem of Line Integrals. Use a computer algebra system to verify your results.

$$\int_C \frac{2x}{(x^2+y^2)^2} dx + \frac{2y}{(x^2+y^2)^2} dy \quad f(x,y) = \frac{-1}{x^2+y^2} \quad (\text{potential function})$$

$$\int_C \vec{F} \cdot d\vec{r} = f(-1,3) - f(13,3) = \frac{-1}{1+9} - \frac{-1}{169+9} = \frac{-42}{445}$$

C: circle $(x-6)^2 + (y-3)^2 = 49$ clockwise from (13,3) to (-1,3)

27. Verify Green's Theorem by setting up and evaluating both integrals

$$\int_C y^2 dx + x^2 dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

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for the path C: square with vertices (0,0), (4,0), (4,4), (0,4).

28. Use Green's Theorem to evaluate the integral

$$\int_C (y-x)dx + (5x-y)dy = \iint_D 5 - 1 dA = 4 \int_0^{19} x - (x^2 - 18x) dx = 4 \int_0^{19} 19x - x^2 dx = 4 \left[\frac{19x^2}{2} - \frac{x^3}{3} \right]_0^{19}$$

for the path C: boundary of the region lying between the graphs of $y = x$ and $y = x^2 - 18x$

29. Use Green's Theorem to evaluate the integral

$$\int_C 13xy dx + (x+y) dy = \iint_D (1 - 13x) dA = \int_{-13}^{13} \int_0^{169-x^2} (1 - 13x) dy dx = \int_{-13}^{13} (1 - 13x)(169 - x^2) dx$$

for the path C: boundary of the region lying between the graphs of $y = 0$ and $y = 169 - x^2$.

$$\int_{-13}^{13} + 13x^3 - x^2 - 13x + 169 dx \quad \text{drop the odd terms}$$

$$= \int_{-13}^{13} -x^2 + 169 dx = \left[\frac{-x^3}{3} + 169x \right]_{-13}^{13}$$

$$= 2 \left(\frac{-13^3}{3} + 13^3 \right) = \frac{-4 \cdot 13^3}{3} = \frac{8788}{3}$$

30. Use Green's Theorem to evaluate the integral

$$\int_C (x^2 - y^2) dx + 10xy dy = \iint_D 10y - (-2y) dA = \iint_D 12y dA = \int_0^{2\pi} \int_0^3 12r^2 \sin \theta r dr d\theta$$

for the path $C: x^2 + y^2 = 9$.

$$\rightarrow [4r^3]_0^3 [-\cos \theta]_0^{2\pi} = 0 \quad \text{since } \int_0^{2\pi} \cos \theta d\theta = 0$$

31. Use Green's Theorem to evaluate the integral

$$\int_C 8xy dx + 8(x+y) dy = 8 \iint_D 1 - x dA = 8(\pi 4^2 - \pi 1^2) - \int_0^{2\pi} \int_0^4 r^2 \cos \theta dr d\theta = 8 \cdot 15\pi = 120\pi$$

for C : boundary of the region lying between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$.

32. Use Green's Theorem to calculate the work done by the force \vec{F} on a particle that is moving counterclockwise around the closed path C .

$$\begin{aligned} \vec{F}(x, y) &= 4xy\hat{i} + (x+y)\hat{j} & \int_C \vec{F} \cdot d\vec{r} = \int 4xy dx + (x+y)dy = \iint 1 - 4x dA \\ C: x^2 + y^2 &= 9 & = \pi 3^2 - 4 \int_0^{2\pi} \int_0^3 r^2 \cos \theta dr d\theta = 9\pi \quad \text{since } \int_0^{2\pi} \cos \theta d\theta = 0 \end{aligned}$$

33. Find the rectangular equation for the surface by eliminating parameters from the vector-valued function. Identify the surface.

$$\begin{aligned} \vec{r}(u, v) &= u\hat{i} + v\hat{j} + \frac{v}{9}\hat{k} & x = u & \quad 9z - v = 0 \quad \text{a plane through} \\ & & y = v & \quad (0, 0, 0) \text{ with normal vector} \\ & & z = \frac{v}{9} = \frac{y}{9} & \quad \langle 0, -1, 9 \rangle \end{aligned}$$

34. Find a vector-valued function whose graph is the cylinder $x^2 + y^2 = 4$.

$$\vec{r}(uv) = 2\cos u \hat{i} + 2\sin u \hat{j} + v\hat{k}$$

- 35.

Find a vector-valued function whose graph is the ellipsoid $\frac{x^2}{49} + \frac{y^2}{100} + \frac{z^2}{81} = 1$.

From the unit sphere in spherical coordinates

$$x = 7\sin\phi\cos\theta, y = 10\sin\phi\sin\theta, z = 9\cos\phi$$

start with $x^2 + y^2 + z^2 = 1$
replace x with $\frac{x}{7}$
 y with $\frac{y}{10}$
 z with $\frac{z}{9}$

36. Write a set of parametric equations for the surface of revolution obtained by revolving the graph of the function about the given axis.

$$y = \frac{x}{7}, \quad 0 \leq x \leq 21 \quad \text{x-axis}$$

$$x = t$$

$$y = \frac{t}{7} \cos \theta$$

$$z = \frac{t}{7} \sin \theta$$

$$\begin{aligned} \downarrow \\ r(\theta, \phi) &= \\ \langle 7\sin\phi\cos\theta, 10\sin\phi\sin\theta, \\ 9\cos\phi \rangle \end{aligned}$$

37. Find an equation of the tangent plane to the surface represented by the vector-valued function at the given point. Normal vector for the tangent plane

$$\vec{r}(u, v) = (10u + v)\hat{i} + (u - v)\hat{j} + v\hat{k}, \quad (6, -6, 6) \text{ is } \vec{r}_u \times \vec{r}_v.$$

$$\vec{r}_u = \langle 10, 1, 0 \rangle, \quad \vec{r}_v = \langle 1, -1, 1 \rangle, \quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i} - 10\hat{j} - 11\hat{k} \Rightarrow (x-6)-10(y+6)-11(z-6)=0$$

38. Find the area of the surface over the given region. Use a computer algebra system to verify your results. $\vec{r}_u = \langle 8\cos u \cos v, 8\cos u \sin v, -8\sin u \rangle$

$$\text{The sphere, } \vec{r}_v = \langle -8\sin u \sin v, 8\sin u \cos v, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = 64 \langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \cos^2 v + \sin u \cos u \sin^2 v \rangle$$

$$\vec{r}(u, v) = 8\sin u \cos v\hat{i} + 8\sin u \sin v\hat{j} + 8\cos u\hat{k}, \quad 0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$$

$$\vec{r}_u \times \vec{r}_v = 64 \sin u \langle \sin u \cos v, \sin u \sin v, \cos u \rangle \text{ so } \|\vec{r}_u \times \vec{r}_v\| = 64 |\sin u| \Rightarrow \int_0^{\pi} \int_0^{2\pi} 64 \sin u \, dv \, du = 256 \cdot 2\pi = 256\pi$$

39. Find the area of the surface over the given region. Use a computer algebra system to verify your results.

$$\vec{r}_u = \langle 3\cos v, 3\sin v, 1 \rangle$$

$$\text{The part of the cone, } \vec{r}_v = \langle -3u \sin v, 3u \cos v, 0 \rangle$$

$$\vec{r}(u, v) = 3u \cos v\hat{i} + 3u \sin v\hat{j} + u\hat{k}, \quad \vec{r}_u \times \vec{r}_v = \langle -3u \cos v, -3u \sin v, 9u \cos^2 v + 9u \sin^2 v \rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = 3\sqrt{u^2 + 9u^2} = 3\sqrt{10}u$$

where $0 \leq u \leq 8$ and $0 \leq v \leq 2\pi$.

$$\int_0^8 \int_0^{2\pi} 3\sqrt{10}u \, dv \, du = 6\pi\sqrt{10} \left[\frac{u^2}{2} \right]_0^8 = 192\sqrt{10}\pi$$

40. Evaluate $\iint_S (x-7y+z)dS$, where

$$dS = \sqrt{1+g_x^2+g_y^2} \, dA = \sqrt{1+1^2+0} \, dA = \sqrt{2} \, dA$$

$$\int_0^{\sqrt{2}} \int_0^{\sqrt{14}} (x-7y+14-x) \, dx \, dy = \int_0^{\sqrt{2}} \int_0^{\sqrt{14}} (14-7y) \, dx \, dy = 14 \int_0^{\sqrt{2}} 14-7y \, dy = 7 \cdot 14 \left[2y - \frac{y^2}{2} \right]_0^{\sqrt{2}} = -5 \cdot 7 \cdot 14^2 \sqrt{2}$$

41. Evaluate $\iint_S f(x, y)dS$, where

$$f(x, y) = y+6 \quad \vec{r}_u = \langle 1, 0, 0 \rangle \quad \vec{r}_u \times \vec{r}_v = \langle 0, -\frac{1}{10}, 1 \rangle$$

$$\vec{r}_v = \langle 0, 1, \frac{1}{10} \rangle$$

$$\int_0^1 \int_0^{10} (\sqrt{1+0^2+\frac{1}{100}}) \, dS = \sqrt{1.01} \left[\frac{v^2}{2} + 6v \right]_0^{10} = \sqrt{1.01} \cdot 110$$

42. Evaluate $\iint_S f(x, y, z)dS$, where

$$f(x, y, z) = x^2 + y^2 + z^2 \quad r(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z \rangle \quad \vec{r}_\theta \times \vec{r}_z = \langle 2\cos\theta, 2\sin\theta, 0 \rangle$$

$$S: x^2 + y^2 = 4, \quad 0 \leq z \leq 4 \quad \vec{r}_\theta = \langle -2\sin\theta, 2\cos\theta, 0 \rangle \quad \|\vec{r}_\theta \times \vec{r}_z\| = 2$$

$$2 \int_0^{2\pi} \int_0^4 4 + z^2 \, dz \, d\theta = 4\pi \left[4z + \frac{z^3}{3} \right]_0^4 = \frac{16 \cdot 28}{3}\pi = \frac{448}{3}\pi$$

43. Find the flux of \bar{F} through S , $\iint_S \bar{F} \cdot \bar{N} dS$, where \bar{N} is the upward unit normal vector to

$$S. \nabla(x+y+z) = \langle 1, 1, 1 \rangle \Rightarrow \bar{N} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \quad dS = \sqrt{1+1+1} dA$$

$$\bar{F}(x, y, z) = 4z\hat{i} - 4\hat{j} + y\hat{k}$$

$S: x+y+z=10$, first octant

$$= \iint_0^{10} \int_0^{10-x} \left(4(10-x-y) - 4 + y \right) dy dx = \iint_0^{10} \int_0^{10-x} 36 - 4x - 3y dy dx$$

$$\iint \langle 4z, -4, y \rangle \cdot \langle 1, 1, 1 \rangle dx dy$$

$$= \int_0^{10} \left[36y - 4xy - \frac{3y^2}{2} \right]_0^{10-x} = \int_0^{10} [110 - 46x + \frac{5x^2}{2}] dx$$

$$= -1100/3$$

44. Find the flux of \bar{F} through S , $\iint_S \bar{F} \cdot \bar{N} dS$, where \bar{N} is the upward unit normal vector to

$$S. G(x, y, z) = z + x^2 + y^2 \quad \nabla G = \langle 2x, 2y, 1 \rangle$$

$$\bar{F}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$$

$$S: z = 36 - x^2 - y^2, z \geq 0$$

$$= \iint 36 + x^2 + y^2 dA = \int_0^{2\pi} \int_0^6 (36 + r^2) r dr d\theta = 2\pi [18r^2 + \frac{r^3}{3}]_0^6 = 1440\pi$$

45. Find the flux of \bar{F} over the closed surface (let \bar{N} be the outward unit normal vector of the surface). *from the divergence theorem*

$$\bar{F}(x, y, z) = 49xy\hat{i} + z^2\hat{j} + yz\hat{k} \quad \iiint 49y + 0 + y dv = 50 \iint_0^6 \int_0^6 y dy dx dz$$

$$S: \text{cube bounded by } x=0, x=6, y=0, y=6, z=0, z=6$$

$$= 50 \cdot 6 \cdot 6 \cdot \frac{6^2}{2} = 32400$$

46. Use the Divergence Theorem to evaluate $\iint_S \bar{F} \cdot \bar{N} dS$. Verify your answer by evaluating

~~the integral as a triple integral.~~

$$F(x, y, z) = 2x\hat{i} - 2y\hat{j} + z^2\hat{k}$$

$$\iiint (2 - 2 + 2z) dv = \int_0^8 \int_0^8 \int_0^8 [z^2]_0^8 dx dy$$

$$= 8^4 = 4096$$

$$S: \text{cube bounded by the planes } x=0, x=8, y=0, y=8, z=0, z=8$$

47. Use the Divergence Theorem to evaluate $\iint_S \bar{F} \cdot \bar{N} dS$ and find the outward flux of \bar{F}

through the surface of the solid bounded by the graphs of the equations. Use a computer-algebra system to verify your results.

$$\bar{F}(x, y, z) = x^2\hat{i} - 2xy\hat{j} + xyz^2\hat{k}$$

$$S: z = \sqrt{100 - x^2 - y^2}, z=0$$

$$\iiint 2x - 2x + 2xyz^2 dv$$

in spherical coordinates

$$2 \int_0^{10} \int_0^{\pi/2} \int_0^{2\pi} xy \cdot p^2 \sin\phi \rho^2 \sin\phi \cos\phi \sin\theta \cos\theta d\theta d\phi dp$$

$$\begin{aligned} u &= \sin\phi \\ du &= \cos\phi d\phi \\ v &= \sin\theta \\ dv &= \cos\theta d\theta \end{aligned}$$

$$2 \int_0^{10} \int_0^{\pi/2} \int_0^{2\pi} p^5 \rho^2 \sin^3\phi \cos\phi \sin\theta \cos\theta d\theta d\phi dp$$

$$2 \int_0^{10} \rho^5 d\rho \int_0^1 u^3 du \int_0^{\pi} v dv = \frac{2 \cdot 10^6}{6} \cdot \frac{1}{4} \cdot 0 = 0$$

48. Use the Divergence Theorem to evaluate $\iint_S \bar{F} \cdot \bar{N} dS$ and find the outward flux of \bar{F} through the surface of the solid bounded by the graphs of the equations. Use a computer algebra system to verify your results. From the divergence theorem

$$\bar{F}(x, y, z) = xyz\hat{j}$$

$$S: x^2 + y^2 = 25, z = 0, z = 7$$

$\iiint_S xz dV$ in cylindrical coordinates

$$\int_0^{2\pi} \int_0^5 \int_0^7 r \cos \theta z r dr dz d\theta = \left[\frac{r^3}{3} \right]_0^7 \left[\sin \theta \right]_0^{2\pi} \left[\frac{z^2}{2} \right]_0^5 = 0$$

49. Use the Divergence Theorem to evaluate $\iint_S \bar{F} \cdot \bar{N} dS$ and find the outward flux of \bar{F}

through the surface of the solid bounded by the graphs of the equations. Use a computer algebra system to verify your results. from the divergence theorem

$$\bar{F}(x, y, z) = xy\hat{i} + 100y\hat{j} + xz\hat{k}$$

$$S: x^2 + y^2 + z^2 = 100$$

$$\iiint_S (y + 100 + x) dV = 100 \cdot \frac{4}{3} \pi 10^3 + \int_0^{10\pi} \int_0^{\pi} \int_0^{2\pi} (\rho \sin \phi \cos \theta + \rho \sin \phi \sin \theta) \rho^2 \sin \phi d\rho d\phi d\theta$$

$$= \frac{400000\pi}{3} + \int_0^{10\pi} \int_0^{\pi} \int_0^{2\pi} \rho^3 \sin^2 \phi (\cos \theta + \sin \theta) d\theta d\phi d\rho = \boxed{\frac{400000\pi}{3}}$$

50. Evaluate $\iint_S \text{curl } \bar{F} \cdot \bar{N} dS$ where S is the closed surface of the solid bounded by the

graphs of $z = 16 - y^2$, $z = 0$, $x = 0$, and $x = 25$.

$$\bar{F}(x, y, z) = xy \cos z \hat{i} + yz \sin x \hat{j} + xyz \hat{k}$$

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51. Find the curl of the vector field $\bar{F}(x, y, z) = (10y - z)\hat{i} + xyz\hat{j} + 6e^z\hat{k}$

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52. Use Stokes's Theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$. Use a computer algebra system to verify your results. Note: C is oriented counterclockwise as viewed from above.

$$\bar{F}(x, y, z) = 6y\hat{i} + 12z\hat{j} + x\hat{k}$$

C : triangle with vertices $(0, 0, 0), (0, 6, 0), (1, 1, 1)$

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53. Use Stokes's Theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$. Use a computer algebra system to verify your results. Note: C is oriented counterclockwise as viewed from above.

$$\bar{F}(x, y, z) = 36xz\hat{i} + y\hat{j} + 36xy\hat{k}$$

$$S: z = 100 - x^2 - y^2, z \geq 0$$

OMIT