1. $\displaystyle\int x^n \, dx = \frac{x^{n+1}}{n+1}$ $\qquad (n \neq -1)$

2. $\displaystyle\int \frac{dx}{x} = \log_e |x|$ $\qquad (x \neq 0)$

3. $\displaystyle\int \sin x \, dx = -\cos x$

4. $\displaystyle\int \cos x \, dx = \sin x$

5. $\displaystyle\int \tan x \, dx = -\log_e |\cos x|$

6. $\displaystyle\int \cot x \, dx = \log_e |\sin x|$

7. $\displaystyle\int \frac{dx}{\cos^3 x} = \tan x$

8. $\displaystyle\int \frac{dx}{\sin^2 x} = -\cot x$

9. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\frac{x}{a}$ \qquad $(a \neq 0)$

10. $\displaystyle\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} = \frac{1}{2a}\log_e\frac{a + x}{a - x}$ \qquad $(|x| < a)$

11. $\displaystyle\int \frac{dx}{x^2 - a^2} = -\frac{1}{a}\coth^{-1}\frac{x}{a} = \frac{1}{2a}\log_e\frac{x - a}{x + a}$ \qquad $(|x| > a)$

12. $\displaystyle\int e^x\, dx = e^x$

13. $\displaystyle\int a^x\, dx = \frac{a^x}{\log_e a}$ \qquad $(a > 0,\ a \neq 1)$

14. $\displaystyle\int \sinh x\, dx = \cosh x$

15. $\displaystyle\int \cosh x\, dx = \sinh x$

16. $\displaystyle\int \tanh x\, dx = \log_e \cosh x$

17. $\displaystyle\int \coth x\, dx = \log_e |\sinh x|$

18. $\displaystyle\int \frac{dx}{\cosh^2 x} = \tanh x$

19. $\displaystyle\int \frac{dx}{\sinh^2 x} = -\coth x$

20. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a}$ \qquad $(a \neq 0)$

21. $\displaystyle\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\frac{x}{a} = \log_e |x + \sqrt{a^2 + x^2}| + C_1$

22. $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} = \log_e |x + \sqrt{x^2 - a^2}| + C_1$

(a) Integrals containing $ax + b$ $(a \neq 0)$

1. $\displaystyle \int (ax + b)^n \, dx = \frac{1}{a(n + 1)} (ax + b)^{n+1}$ $(n \neq -1)$

2. $\displaystyle \int \frac{dx}{ax + b} = \frac{1}{a} \log_e (ax + b)$

3. $\displaystyle \int x(ax + b)^n \, dx = \frac{1}{a^2(n + 2)} (ax + b)^{n+2}$
$$- \frac{b}{a^2(n + 1)} (ax + b)^{n+1} \qquad (n \neq -1, -2)$$

4. $\displaystyle \int x^m(ax + b)^n \, dx$
$$= \frac{1}{a(m + n + 1)} \left[x^m(ax + b)^{n+1} - mb \int x^{m-1}(ax + b)^n \, dx \right]$$
$$= \frac{1}{m + n + 1} \left[x^{m+1}(ax + b)^n + nb \int x^m(ax + b)^{n-1} \, dx \right]$$
$$(m > 0, \ m + n + 1 \neq 0)$$

5. $\displaystyle \int \frac{x \, dx}{ax + b} = \frac{x}{a} - \frac{b}{a^2} \log_e (ax + b)$

6. $\displaystyle \int \frac{x \, dx}{(ax + b)^2} = \frac{b}{a^2(ax + b)} + \frac{1}{a^2} \log_e (ax + b)$

7. $\displaystyle \int \frac{x \, dx}{(ax + b)^3} = \frac{b}{2a^2(ax + b)^2} - \frac{1}{a^2(ax + b)}$

8. $\displaystyle \int \frac{x \, dx}{(ax + b)^n} = \frac{1}{a^2} \left[\frac{b}{(n - 1)(ax + b)^{n-1}} - \frac{1}{(n - 2)(ax + b)^{n-2}} \right]$
$$(n \neq 1, 2)$$

9. $\displaystyle\int \frac{x^2\,dx}{ax+b} = \frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2 - 2b(ax+b) + b^2\log_e(ax+b)\right]$

10. $\displaystyle\int \frac{x^2\,dx}{(ax+b)^2} = \frac{1}{a^3}\left[(ax+b) - 2b\log_e(ax+b) - \frac{b^2}{ax+b}\right]$

11. $\displaystyle\int \frac{x^2\,dx}{(ax+b)^3} = \frac{1}{a^3}\left[\log_e(ax+b) + \frac{2b}{ax+b} - \frac{b^2}{2(ax+b)^2}\right]$

12. $\displaystyle\int x^2(ax+b)^n\,dx$

$$= \frac{1}{a^3}\left[\frac{(ax+b)^{n+3}}{n+3} - 2b\frac{(ax+b)^{n+2}}{n+2} + b^2\frac{(ax+b)^{n+1}}{n+1}\right]$$

$$(n \neq -1, -2, -3)$$

13. $\displaystyle\int \frac{dx}{x(ax+b)} = \frac{1}{b}\log_e\frac{x}{ax+b}$

14. $\displaystyle\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} - \frac{1}{b^2}\log_e\frac{ax+b}{x}$

15. $\displaystyle\int \frac{dx}{x(ax+b)^3} = \frac{1}{b^3}\left[\frac{1}{2}\left(\frac{ax+2b}{ax+b}\right)^2 + \log_e\frac{x}{ax+b}\right]$

16. $\displaystyle\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2}\log_e\frac{ax+b}{x}$

17. $\displaystyle\int \frac{dx}{x^2(ax+b)^2} = -\frac{b+2ax}{b^2x(ax+b)} + \frac{2a}{b^3}\log_e\frac{ax+b}{x}$

18. $\displaystyle\int \frac{dx}{x^3(ax+b)} = \frac{2ax-b}{2b^2x^2} + \frac{a^2}{b^3}\log_e\frac{x}{ax+b}$

Let $ax + b \equiv X$ $(a \neq 0)$. Then

19. $\displaystyle\int \frac{x^3\,dx}{X} = \frac{1}{a^4}\left(\frac{X^3}{3} - \frac{3bX^2}{2} + 3b^2X - b^3\log_e X\right)$

20. $\displaystyle\int \frac{x^3\,dx}{X^2} = \frac{1}{a^4}\left(\frac{X^2}{2} - 3bX + 3b^2\log_e X + \frac{b^3}{X}\right)$

21. $\displaystyle\int \frac{x^3\,dx}{X^3} = \frac{1}{a^4}\left(X - 3b\log_e X - \frac{3b^2}{X} + \frac{b^3}{2X^2}\right)$

22. $\int \dfrac{x^3\, dx}{X^4} = \dfrac{1}{a^4}\left(\log_e X + \dfrac{3b}{X} - \dfrac{3b^2}{2X^2} + \dfrac{b^3}{3X^3}\right)$

23. $\int \dfrac{x^3\, dx}{X^n}$

$= \dfrac{1}{a^4}\left[\dfrac{-1}{(n-4)X^{n-4}} + \dfrac{3b}{(n-3)X^{n-3}} - \dfrac{3b^2}{(n-2)X^{n-2}} + \dfrac{b^3}{(n-1)X^{n-1}}\right]$

$(n \neq 1, 2, 3, 4)$

24. $\int \dfrac{dx}{xX^n} = -\dfrac{1}{b^n}\left[\log_e \dfrac{X}{x} - \sum_{k=1}^{n-1}\binom{n-1}{k}\dfrac{(-a)^k x^k}{kX^k}\right], \quad (n \geq 1)$

25. $\int \dfrac{dx}{x^2 X^3} = -a\left[\dfrac{1}{2b^2X^2} + \dfrac{2}{b^3X} + \dfrac{1}{ab^3x} - \dfrac{3}{b^4}\log_e \dfrac{X}{x}\right]$

26. $\int \dfrac{dx}{x^2 X^n} = -\dfrac{1}{b^{n+1}}\left[-\sum_{k=2}^{n}\binom{n}{k}\dfrac{(-a)^k x^{k-1}}{(k-1)X^{k-1}} + \dfrac{X}{x} - na\log_e \dfrac{X}{x}\right]$

$(n \geq 2)$

27. $\int \dfrac{dx}{x^3 X^2} = -\dfrac{1}{b^4}\left[3a^2\log_e \dfrac{X}{x} + \dfrac{a^3x}{X} + \dfrac{X^2}{2x^2} - \dfrac{3aX}{x}\right]$

28. $\int \dfrac{dx}{x^3 X^3} = -\dfrac{1}{b^5}\left[6a^2\log_e \dfrac{X}{x} + \dfrac{4a^3x}{X} - \dfrac{a^4x^2}{2X^2} + \dfrac{X^2}{2x^2} - \dfrac{4aX}{x}\right]$

29. $\int \dfrac{dx}{x^3 X^n} = -\dfrac{1}{b^{n+2}}\left[-\sum_{k=3}^{n+1}\binom{n+1}{k}\dfrac{(-a)^k x^{k-2}}{(k-2)X^{k-2}} + \dfrac{a^2X^2}{2x^2}\right.$

$\left. - \dfrac{(n+1)aX}{x} + \dfrac{n(n+1)a^2}{2}\log_e \dfrac{X}{x}\right] \quad (n \geq 3)$

30. $\int \dfrac{dx}{x^m X^n} = -\dfrac{1}{b^{m+n-1}}\sum_{k=0}^{m+n-2}\binom{m+n-2}{k}\dfrac{X^{m-k-1}(-a)^k}{(m-k-1)x^{m-k-1}}$

$\left[\text{terms with } (m-k-1) = 0 \text{ are replaced by}\right.$

$\left.\binom{m+n-2}{m-1}(-a)^{m-1}\log_e \dfrac{X}{x}\right]$

(b) Integrals containing $ax + b$ and $cx + d$ $\qquad (a \neq 0, c \neq 0)$

31. $\int \dfrac{ax+b}{cx+d}\, dx = \dfrac{a}{c}x + \dfrac{bc-ad}{c^2}\log_e (cx+d)$

32. $\displaystyle\int \frac{dx}{(ax + b)(cx + d)} = \frac{1}{bc - ad} \log_e \frac{cx + d}{ax + b}$ $(bc - ad \neq 0)$

33. $\displaystyle\int \frac{x\,dx}{(ax + b)(cx + d)}$

$$= \frac{1}{bc - ad}\left[\frac{b}{a}\log(ax + b) - \frac{d}{c}\log_e(cx + d)\right] \qquad (bc - ad \neq 0)$$

34. $\displaystyle\int \frac{dx}{(ax + b)^2(cx + d)}$

$$= \frac{1}{bc - ad}\left[\frac{1}{ax + b} + \frac{c}{bc - ad}\log_e\frac{cx + d}{ax + b}\right] \qquad (bc - ad \neq 0)$$

(c) Integrals containing $a + x$ and $b + x$ $(a \neq b)$

35. $\displaystyle\int \frac{x\,dx}{(a + x)(b + x)^2} = \frac{b}{(a - b)(b + x)} - \frac{a}{(a - b)^2}\log_e\frac{a + x}{b + x}$

36. $\displaystyle\int \frac{x^2\,dx}{(a + x)(b + x)^2} = \frac{b^2}{(b - a)(b + x)} + \frac{a^2}{(b - a)^2}\log_e(a + x)$

$$+ \frac{b^2 - 2ab}{(b - a)^2}\log_e(b + x)$$

37. $\displaystyle\int \frac{dx}{(a + x)^2(b + x)^2}$

$$= \frac{-1}{(a - b)^2}\left(\frac{1}{a + x} + \frac{1}{b + x}\right) + \frac{2}{(a - b)^3}\log_e\frac{a + x}{b + x}$$

38. $\displaystyle\int \frac{x\,dx}{(a + x)^2(b + x)^2}$

$$= \frac{1}{(a - b)^2}\left(\frac{a}{a + x} + \frac{b}{b + x}\right) + \frac{a + b}{(a - b)^3}\log_e\frac{a + x}{b + x}$$

39. $\displaystyle\int \frac{x^2\,dx}{(a + x)^2(b + x)^2}$

$$= \frac{-1}{(a - b)^2}\left(\frac{a^2}{a + x} + \frac{b^2}{b + x}\right) + \frac{2ab}{(a - b)^3}\log_e\frac{a + x}{b + x}$$

(d) Integrals containing $ax^2 + bx + c$ $\quad (a \neq 0)$

40. $\displaystyle\int \frac{dx}{ax^2 + bx + c}$

$$= \begin{cases} \dfrac{1}{\sqrt{b^2 - 4ac}} \log_e \dfrac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} & (b^2 > 4ac) \\[2ex] \dfrac{2}{\sqrt{4ac - b^2}} \arctan \dfrac{2ax + b}{\sqrt{4ac - b^2}} & (b^2 < 4ac) \\[2ex] -\dfrac{2}{2ax + b} & (b^2 = 4ac) \end{cases}$$

In numbers 41 through 47, let $b^2 - 4ac \neq 0$.

41. $\displaystyle\int \frac{dx}{(ax^2 + bx + c)^2} = \frac{2ax + b}{(4ac - b^2)(ax^2 + bx + c)}$
$$+ \frac{2a}{(4ac - b^2)} \int \frac{dx}{ax^2 + bx + c}$$

42. $\displaystyle\int \frac{dx}{(ax^2 + bx + c)^{n+1}} = \frac{2ax + b}{n(4ac - b^2)(ax^2 + bx + c)^n}$
$$+ \frac{2(2n - 1)a}{n(4ac - b^2)} \int \frac{dx}{(ax^2 + bx + c)^n}$$

43. $\displaystyle\int \frac{x\,dx}{ax^2 + bx + c} = \frac{1}{2a} \log_e (ax^2 + bx + c) - \frac{b}{2a} \int \frac{dx}{ax^2 + bx + c}$

44. $\displaystyle\int \frac{x^2\,dx}{ax^2 + bx + c} = \frac{x}{a} - \frac{b}{2a^2} \log_e (ax^2 + bx + c)$
$$+ \frac{b^2 - 2ac}{2a^2} \int \frac{dx}{ax^2 + bx + c}$$

45. $\displaystyle\int \frac{x^n\,dx}{ax^2 + bx + c} = \frac{x^{n-1}}{(n - 1)a} - \frac{c}{a} \int \frac{x^{n-2}\,dx}{ax^2 + bx + c}$
$$- \frac{b}{a} \int \frac{x^{n-1}\,dx}{ax^2 + bx + c} \qquad (n \neq 1)$$

46. $\displaystyle\int \frac{x\,dx}{(ax^2 + bx + c)^{n+1}} = \frac{-(2c + bx)}{n(4ac - b^2)(ax^2 + bx + c)^n}$
$$- \frac{b(2n - 1)}{n(4ac - b^2)} \int \frac{dx}{(ax^2 + bx + c)^n}$$

47. $\displaystyle\int \frac{x^m\,dx}{(ax^2 + bx + c)^{n+1}} = -\frac{x^{m-1}}{a(2n-m+1)(ax^2+bx+c)^n}$

$$-\frac{n-m+1}{2n-m+1}\cdot\frac{b}{a}\int\frac{x^{m-1}\,dx}{(ax^2+bx+c)^{n+1}}$$

$$+\frac{m-1}{2n-m+1}\cdot\frac{c}{a}\int\frac{x^{m-2}\,dx}{(ax^2+bx+c)^{n+1}} \qquad (m \neq 2n+1)$$

Let $ax^2 + bx + c \equiv X\ (a \neq 0)$. Then

48. $\displaystyle\int \frac{x^{2n-1}\,dx}{X^n} = \frac{1}{a}\int\frac{x^{2n-3}\,dx}{X^{n-1}} - \frac{c}{a}\int\frac{x^{2n-3}\,dx}{X^n} - \frac{b}{a}\int\frac{x^{2n-2}\,dx}{X^n}$

49. $\displaystyle\int \frac{dx}{xX} = \frac{1}{2c}\log_e\frac{x^2}{X} - \frac{b}{2c}\int\frac{dx}{X}$

50. $\displaystyle\int \frac{dx}{xX^n} = \frac{1}{2c(n-1)X^{n-1}} - \frac{b}{2c}\int\frac{dx}{X^n} + \frac{1}{c}\int\frac{dx}{xX^{n-1}}$

51. $\displaystyle\int \frac{dx}{x^2X} = \frac{b}{2c^2}\log_e\frac{X}{x^2} - \frac{1}{cx} + \left(\frac{b^2}{2c^2} - \frac{a}{c}\right)\int\frac{dx}{X}$

52. $\displaystyle\int \frac{dx}{x^mX^n} = -\frac{1}{(m-1)cx^{m-1}X^{n-1}} - \frac{(2n+m-3)a}{(m-1)c}\int\frac{dx}{x^{m-2}X^n}$

$$-\frac{(n+m-2)b}{(m-1)c}\int\frac{dx}{x^{m-1}X^n} \qquad (m > 1)$$

53. $\displaystyle\int \frac{dx}{(fx+g)X} = \frac{1}{2(cf^2 - gbf + g^2a)}\left[f\log_e\frac{(fx+g)^2}{X}\right]$

$$+\frac{2ga - bf}{2(cf^2 - gbf + g^2a)}\int\frac{dx}{X}$$

(e) Integrals containing $a^2 \pm x^2$, with

$$X \equiv a^2 + x^2 \qquad Y \equiv \arctan\frac{x}{a}$$

or

$$X \equiv a^2 - x^2 \qquad Y \equiv \tanh^{-1}\frac{x}{a} \equiv \begin{cases} \dfrac{1}{2}\log_e\dfrac{a+x}{a-x} & (|x| < a) \\[2mm] \dfrac{1}{2}\log_e\dfrac{x+a}{x-a} & (|x| > a) \end{cases}$$

Where $\pm$ or $\mp$ appears in a formula, the upper sign refers to $X \equiv a^2 + x^2$ and the lower sign to $X \equiv a^2 - x^2$ ($a \neq 0$).

54. $\displaystyle\int \frac{dx}{X} = \frac{1}{a} Y$

55. $\displaystyle\int \frac{dx}{X^2} = \frac{x}{2a^2 X} + \frac{1}{2a^3} Y$

56. $\displaystyle\int \frac{dx}{X^3} = \frac{x}{4a^2 X^2} + \frac{3x}{8a^4 X} + \frac{3}{8a^5} Y$

57. $\displaystyle\int \frac{dx}{X^{n+1}} = \frac{x}{2na^2 X^n} + \frac{2n-1}{2na^2} \int \frac{dx}{X^n}$

58. $\displaystyle\int \frac{x\,dx}{X} = \pm \frac{1}{2} \log_e X$

59. $\displaystyle\int \frac{x\,dx}{X^2} = \mp \frac{1}{2X}$

60. $\displaystyle\int \frac{x\,dx}{X^3} = \mp \frac{1}{4X^2}$

61. $\displaystyle\int \frac{x\,dx}{X^{n+1}} = \mp \frac{1}{2nX^n} \qquad (n \neq 0)$

62. $\displaystyle\int \frac{x^2\,dx}{X} = \pm x \mp aY$

63. $\displaystyle\int \frac{x^2\,dx}{X^2} = \mp \frac{x}{2X} \pm \frac{1}{2a} Y$

64. $\displaystyle\int \frac{x^2\,dx}{X^3} = \mp \frac{x}{4X^2} \pm \frac{x}{8a^2 X} \pm \frac{1}{8a^3} Y$

65. $\displaystyle\int \frac{x^2\,dx}{X^{n+1}} = \mp \frac{x}{2nX^n} \pm \frac{1}{2n} \int \frac{dx}{X^n} \qquad (n \neq 0)$

66. $\displaystyle\int \frac{x^3\,dx}{X} = \pm \frac{x^2}{2} - \frac{a^2}{2} \log_e X$

67. $\displaystyle\int \frac{x^3\,dx}{X^2} = \frac{a^2}{2X} + \frac{1}{2} \log_e X$

68. $\displaystyle\int \frac{x^3\,dx}{X^3} = -\frac{1}{2X} + \frac{a^2}{4X^2}$

69. $\int \dfrac{x^3\, dx}{X^{n+1}} = -\dfrac{1}{2(n-1)X^{n-1}} + \dfrac{a^2}{2nX^n}$ $\qquad (n > 1)$

70. $\int \dfrac{dx}{xX} = \dfrac{1}{2a^2} \log_e \dfrac{x^2}{X}$

71. $\int \dfrac{dx}{xX^2} = \dfrac{1}{2a^2X} + \dfrac{1}{2a^4} \log_e \dfrac{x^2}{X}$

72. $\int \dfrac{dx}{xX^3} = \dfrac{1}{4a^2X^2} + \dfrac{1}{2a^4X} + \dfrac{1}{2a^6} \log_e \dfrac{x^2}{X}$

73. $\int \dfrac{dx}{x^2X} = -\dfrac{1}{a^2x} \mp \dfrac{1}{a^3}\, Y$

74. $\int \dfrac{dx}{x^2X^2} = -\dfrac{1}{a^4x} \mp \dfrac{x}{2a^4X} \mp \dfrac{3}{2a^5}\, Y$

75. $\int \dfrac{dx}{x^2X^3} = -\dfrac{1}{a^6x} \mp \dfrac{x}{4a^4X^2} \mp \dfrac{7x}{8a^6X} \mp \dfrac{15}{8a^7}\, Y$

76. $\int \dfrac{dx}{x^3X} = -\dfrac{1}{2a^2x^2} \mp \dfrac{1}{2a^4} \log_e \dfrac{x^2}{X}$

77. $\int \dfrac{dx}{x^3X^2} = -\dfrac{1}{2a^4x^2} \mp \dfrac{1}{2a^4X} \mp \dfrac{1}{a^6} \log_e \dfrac{x^2}{X}$

78. $\int \dfrac{dx}{x^3X^3} = -\dfrac{1}{2a^6x^2} \mp \dfrac{1}{a^6X} \mp \dfrac{1}{4a^4X^2} \mp \dfrac{3}{2a^8} \log_e \dfrac{x^2}{X}$

79. $\int \dfrac{dx}{(b + cx)X} = \dfrac{1}{a^2c^2 \pm b^2}\left[c \log_e (b + cx) - \dfrac{c}{2} \log_e X \pm \dfrac{b}{a}\, Y \right]$

(f) *Integrals containing* $a^3 \pm x^3$, *with*

$X \equiv a^3 \pm x^3$ $\qquad (a \neq 0)$

Where $\pm$ or $\mp$ appears in a formula, the upper sign refers to $X \equiv a^3 + x^3$ and the lower sign to $X \equiv a^3 - x^3$.

80. $\int \dfrac{dx}{X} = \pm \dfrac{1}{6a^2} \log_e \dfrac{(a \pm x)^2}{a^2 \mp ax + x^2} + \dfrac{1}{a^2\sqrt{3}} \arctan \dfrac{2x \mp a}{a\sqrt{3}}$

81. $\int \dfrac{dx}{X^2} = \dfrac{x}{3a^3X} + \dfrac{2}{3a^3} \int \dfrac{dx}{X}$

82. $\displaystyle\int \frac{x\,dx}{X} = \frac{1}{6a}\log_e \frac{a^2 \mp ax + x^2}{(a \pm x)^2} \pm \frac{1}{a\sqrt{3}}\arctan\frac{2x \mp a}{a\sqrt{3}}$

83. $\displaystyle\int \frac{x\,dx}{X^2} = \frac{x^2}{3a^3 X} + \frac{1}{3a^3}\int \frac{x\,dx}{X}$

84. $\displaystyle\int \frac{x^2\,dx}{X} = \pm\frac{1}{3}\log_e X$

85. $\displaystyle\int \frac{x^2\,dx}{X^2} = \mp\frac{1}{3X}$

86. $\displaystyle\int \frac{x^3\,dx}{X} = \pm x \mp a^3 \int \frac{dx}{X}$

87. $\displaystyle\int \frac{x^3\,dx}{X^2} = \mp\frac{x}{3X} \pm \frac{1}{3}\int \frac{dx}{X}$

88. $\displaystyle\int \frac{dx}{xX} = \frac{1}{3a^3}\log_e \frac{x^3}{X}$

89. $\displaystyle\int \frac{dx}{xX^2} = \frac{1}{3a^3 X} + \frac{1}{3a^6}\log_e \frac{x^3}{X}$

90. $\displaystyle\int \frac{dx}{x^2 X} = -\frac{1}{a^3 x} \mp \frac{1}{a^3}\int \frac{x\,dx}{X}$

91. $\displaystyle\int \frac{dx}{x^2 X^2} = -\frac{1}{a^6 x} \mp \frac{x^2}{3a^6 X} \mp \frac{4}{3a^6}\int \frac{x\,dx}{X}$

92. $\displaystyle\int \frac{dx}{x^3 X} = -\frac{1}{2a^3 x^2} \mp \frac{1}{a^3}\int \frac{dx}{X}$

93. $\displaystyle\int \frac{dx}{x^3 X^2} = -\frac{1}{2a^6 x^2} \mp \frac{x}{3a^6 X} \mp \frac{5}{3a^6}\int \frac{dx}{X}$

(g) Integrals containing $a^4 \pm x^4$ $(a \neq 0)$

94. $\displaystyle\int \frac{dx}{a^4 + x^4} = \frac{1}{4a^3\sqrt{2}}\log_e \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a^3\sqrt{2}}\arctan\frac{ax\sqrt{2}}{a^2 - x^2}$

95. $\displaystyle\int \frac{x\,dx}{a^4 + x^4} = \frac{1}{2a^2}\arctan\frac{x^2}{a^2}$

96. $\displaystyle\int \frac{x^2\,dx}{a^4 + x^4} = -\frac{1}{4a\sqrt{2}}\log_e \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a\sqrt{2}}\arctan\frac{ax\sqrt{2}}{a^2 - x^2}$

97. $\int \dfrac{x^3\,dx}{a^4 + x^4} = \dfrac{1}{4} \log_e (a^4 + x^4)$

98. $\int \dfrac{dx}{a^4 - x^4} = \dfrac{1}{4a^3} \log_e \dfrac{a + x}{a - x} + \dfrac{1}{2a^3} \arctan \dfrac{x}{a}$

99. $\int \dfrac{x\,dx}{a^4 - x^4} = \dfrac{1}{4a^3} \log_e \dfrac{a^2 + x^2}{a^2 - x^2}$

100. $\int \dfrac{x^2\,dx}{a^4 - x^4} = \dfrac{1}{4a} \log_e \dfrac{a + x}{a - x} - \dfrac{1}{2a} \arctan \dfrac{x}{a}$

101. $\int \dfrac{x^3\,dx}{a^4 - x^4} = -\dfrac{1}{4} \log_e (a^4 - x^4)$

(h) Integrals containing $\sqrt{x}$ and $a^2 + b^2x$ $(a, b \neq 0)$

102. $\int \dfrac{\sqrt{x}\,dx}{a^2 + b^2x} = \dfrac{2\sqrt{x}}{b^2} - \dfrac{2a}{b^3} \arctan \dfrac{b\sqrt{x}}{a}$

103. $\int \dfrac{x\sqrt{x}\,dx}{a^2 + b^2x} = \dfrac{2x\sqrt{x}}{3b^2} - \dfrac{2a^2\sqrt{x}}{b^4} + \dfrac{2a^3}{b^5} \arctan \dfrac{b\sqrt{x}}{a}$

104. $\int \dfrac{\sqrt{x}\,dx}{(a^2 + b^2x)^2} = -\dfrac{\sqrt{x}}{b^2(a^2 + b^2x)} + \dfrac{1}{ab^3} \arctan \dfrac{b\sqrt{x}}{a}$

105. $\int \dfrac{x\sqrt{x}\,dx}{(a^2 + b^2x)^2} = \dfrac{2b^2x\sqrt{x} + 3a^2\sqrt{x}}{b^4(a^2 + b^2x)} - \dfrac{3a}{b^5} \arctan \dfrac{b\sqrt{x}}{a}$

106. $\int \dfrac{dx}{(a^2 + b^2x)\sqrt{x}} = \dfrac{2}{ab} \arctan \dfrac{b\sqrt{x}}{a}$

107. $\int \dfrac{dx}{(a^2 + b^2x)x\sqrt{x}} = -\dfrac{2}{a^2\sqrt{x}} - \dfrac{2b}{a^3} \arctan \dfrac{b\sqrt{x}}{a}$

108. $\int \dfrac{dx}{(a^2 + b^2x)^2\sqrt{x}} = \dfrac{\sqrt{x}}{a^2(a^2 + b^2x)} + \dfrac{1}{a^3b} \arctan \dfrac{b\sqrt{x}}{a}$

(i) Integrals containing $\sqrt{x}$ and $a^2 - b^2x > 0$ $(a, b \neq 0)$

109. $\int \dfrac{\sqrt{x}\,dx}{a^2 - b^2x} = -\dfrac{2\sqrt{x}}{b^2} + \dfrac{a}{b^3} \log_e \dfrac{a + b\sqrt{x}}{a - b\sqrt{x}}$

110. $\int \dfrac{x\sqrt{x}\,dx}{a^2 - b^2x} = -\dfrac{2x\sqrt{x}}{3b^2} - \dfrac{2a^2\sqrt{x}}{b^4} + \dfrac{a^3}{b^5} \log_e \dfrac{a + b\sqrt{x}}{a - b\sqrt{x}}$

111. $\displaystyle\int \frac{\sqrt{x}\,dx}{(a^2 - b^2 x)^2} = \frac{\sqrt{x}}{b^2(a^2 - b^2 x)} - \frac{1}{2ab^3} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$

112. $\displaystyle\int \frac{x\sqrt{x}\,dx}{(a^2 - b^2 x)^2} = \frac{-2b^2 x\sqrt{x} + 3a^2\sqrt{x}}{b^4(a^2 - b^2 x)} - \frac{3a}{2b^5} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$

113. $\displaystyle\int \frac{dx}{(a^2 - b^2 x)\sqrt{x}} = \frac{1}{ab} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$

114. $\displaystyle\int \frac{dx}{(a^2 - b^2 x)x\sqrt{x}} = -\frac{2}{a^2\sqrt{x}} + \frac{b}{a^3} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$

115. $\displaystyle\int \frac{dx}{(a^2 - b^2 x)^2\sqrt{x}} = \frac{\sqrt{x}}{a^2(a^2 - b^2 x)} + \frac{1}{2a^3 b} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$

116. $\displaystyle\int \frac{dx}{(a^2 - b^2 x)^2 x\sqrt{x}} = -\frac{2}{a^2(a^2 - b^2 x)\sqrt{x}} + \frac{3b^2\sqrt{x}}{a^4(a^2 - b^2 x)}$
$$+ \frac{3b}{2a^5} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$$

(j) Other integrals containing $\sqrt{x}$ $\qquad (a > \sqrt{x} > 0)$

117. $\displaystyle\int \frac{\sqrt{x}\,dx}{a^4 + x^2} = -\frac{1}{2a\sqrt{2}} \log_e \frac{x + a\sqrt{2x} + a^2}{x - a\sqrt{2x} + a^2} + \frac{1}{a\sqrt{2}} \arctan \frac{a\sqrt{2x}}{a^2 - x}$

118. $\displaystyle\int \frac{dx}{(a^4 + x^2)\sqrt{x}} = \frac{1}{2a^3\sqrt{2}} \log_e \frac{x + a\sqrt{2x} + a^2}{x - a\sqrt{2x} + a^2}$
$$+ \frac{1}{a^3\sqrt{2}} \arctan \frac{a\sqrt{2x}}{a^2 - x}$$

119. $\displaystyle\int \frac{\sqrt{x}\,dx}{a^4 - x^2} = \frac{1}{2a} \log_e \frac{a + \sqrt{x}}{a - \sqrt{x}} - \frac{1}{a} \arctan \frac{\sqrt{x}}{a}$

120. $\displaystyle\int \frac{dx}{(a^4 - x^2)\sqrt{x}} = \frac{1}{2a^3} \log_e \frac{a + \sqrt{x}}{a - \sqrt{x}} + \frac{1}{a^3} \arctan \frac{\sqrt{x}}{a}$

(k) Integrals containing $\sqrt{ax + b}$, with

$$\boxed{X \equiv ax + b \qquad Y \equiv fx + g \qquad \Delta \equiv bf - ag \qquad (a \neq 0)}$$

121. $\displaystyle\int \sqrt{X}\, dx = \frac{2}{3a}\sqrt{X^3}$

122. $\displaystyle\int x\sqrt{X}\, dx = \frac{2(3ax - 2b)\sqrt{X^3}}{15a^2}$

123. $\displaystyle\int x^2\sqrt{X}\, dx = \frac{2(15a^2x^2 - 12abx + 8b^2)\sqrt{X^3}}{105a^3}$

124. $\displaystyle\int \frac{dx}{\sqrt{X}} = \frac{2\sqrt{X}}{a}$

125. $\displaystyle\int \frac{x\, dx}{\sqrt{X}} = \frac{2(ax - 2b)}{3a^2}\sqrt{X}$

126. $\displaystyle\int \frac{x^2\, dx}{\sqrt{X}} = \frac{2(3a^2x^2 - 4abx + 8b^2)\sqrt{X}}{15a^3}$

127. $\displaystyle\int \frac{dx}{x\sqrt{X}} = \begin{cases} -\dfrac{2}{\sqrt{b}}\tanh^{-1}\sqrt{\dfrac{X}{b}} = \dfrac{1}{\sqrt{b}}\log_e \dfrac{\sqrt{X} - \sqrt{b}}{\sqrt{X} + \sqrt{b}} & (b > 0) \\[4mm] \dfrac{2}{\sqrt{-b}}\arctan\sqrt{\dfrac{X}{-b}} & (b < 0) \end{cases}$

128. $\displaystyle\int \frac{\sqrt{X}}{x}\, dx = 2\sqrt{X} + b\int \frac{dx}{x\sqrt{X}}$

129. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{X}}$

130. $\displaystyle\int \frac{\sqrt{X}}{x^2}\, dx = -\frac{\sqrt{X}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{X}}$

131. $\displaystyle\int \frac{dx}{x^n\sqrt{X}} = -\frac{\sqrt{X}}{(n - 1)bx^{n-1}} - \frac{(2n - 3)a}{(2n - 2)b}\int \frac{dx}{x^{n-1}\sqrt{X}} \qquad (n \neq 1)$

132. $\displaystyle\int \sqrt{X^3}\, dx = \frac{2\sqrt{X^5}}{5a}$

133. $\displaystyle\int x\sqrt{X^3}\, dx = \frac{2}{35a^2}(5\sqrt{X^7} - 7b\sqrt{X^5})$

134. $\displaystyle\int x^2\sqrt{X^3}\, dx = \frac{2}{a^3}\left(\frac{\sqrt{X^9}}{9} - \frac{2b\sqrt{X^7}}{7} + \frac{b^2\sqrt{X^5}}{5}\right)$

135. $\displaystyle \int \frac{\sqrt{X^3}}{x}\, dx = \frac{2\sqrt{X^3}}{3} + 2b\sqrt{X} + b^2 \int \frac{dx}{x\sqrt{X}}$

136. $\displaystyle \int \frac{x\, dx}{\sqrt{X^3}} = \frac{2}{a^2}\left(\sqrt{X} + \frac{b}{\sqrt{X}}\right)$

137. $\displaystyle \int \frac{x^2\, dx}{\sqrt{X^3}} = \frac{2}{a^3}\left(\frac{\sqrt{X^3}}{3} - 2b\sqrt{X} - \frac{b^2}{\sqrt{X}}\right)$

138. $\displaystyle \int \frac{dx}{x\sqrt{X^3}} = \frac{2}{b\sqrt{X}} + \frac{1}{b}\int \frac{dx}{x\sqrt{X}} \qquad (b \neq 0)$

139. $\displaystyle \int \frac{dx}{x^2\sqrt{X^3}} = -\frac{1}{bx\sqrt{X}} - \frac{3a}{b^2\sqrt{X}} - \frac{3a}{2b^2}\int \frac{dx}{x\sqrt{X}} \qquad (b \neq 0)$

140. $\displaystyle \int X^{\pm n/2}\, dx = \frac{2X^{(2\pm n)/2}}{a(2 \pm n)} \qquad (n \neq \mp 2)$

141. $\displaystyle \int x X^{\pm n/2}\, dx = \frac{2}{a^2}\left(\frac{X^{(4\pm n)/2}}{4 \pm n} - \frac{bX^{(2\pm n)/2}}{2 \pm n}\right) \qquad (n \neq \mp 2, \mp 4)$

142. $\displaystyle \int x^2 X^{\pm n/2}\, dx = \frac{2}{a^3}\left(\frac{X^{(6\pm n)/2}}{6 \pm n} - \frac{2bX^{(4\pm n)/2}}{4 \pm n} + \frac{b^2 X^{(2\pm n)/2}}{2 \pm n}\right)$

$$(n \neq \mp 2,\ \mp 4,\ \mp 6)$$

143. $\displaystyle \int \frac{X^{n/2}\, dx}{x} = \frac{2X^{n/2}}{n} + b\int \frac{X^{(n-2)/2}}{x}\, dx \qquad (n \neq 0)$

144. $\displaystyle \int \frac{dx}{xX^{n/2}} = \frac{2}{(n-2)bX^{(n-2)/2}} + \frac{1}{b}\int \frac{dx}{xX^{(n-2)/2}} \qquad (n \neq 2, b \neq 0)$

145. $\displaystyle \int \frac{dx}{x^2 X^{n/2}} = -\frac{1}{bxX^{(n-2)/2}} - \frac{na}{2b}\int \frac{dx}{xX^{n/2}} \qquad (b \neq 0)$

146. $\displaystyle \int \frac{dx}{\sqrt{XY}} = \begin{cases} \dfrac{2}{\sqrt{-af}}\arctan\sqrt{-\dfrac{fX}{aY}} & (af < 0) \\[3mm] \dfrac{2}{\sqrt{af}}\tanh^{-1}\sqrt{\dfrac{fX}{aY}} = \dfrac{2}{\sqrt{af}}\log_e\left(\sqrt{aY} + \sqrt{fX}\right) + C_1 \end{cases}$

$$(af > 0)$$

147. $\displaystyle \int \frac{x\, dx}{\sqrt{XY}} = \frac{\sqrt{XY}}{af} - \frac{ag + bf}{2af}\int \frac{dx}{\sqrt{XY}}$

148. $\int \dfrac{dx}{\sqrt{X}\sqrt{Y^3}} = -\dfrac{2\sqrt{X}}{\Delta\sqrt{Y}}$

149. $\int \dfrac{dx}{Y\sqrt{X}} = \begin{cases} \dfrac{2}{\sqrt{-f\Delta}} \arctan \dfrac{f\sqrt{X}}{\sqrt{-f\Delta}} & (f\Delta < 0) \\[3mm] \dfrac{1}{\sqrt{f\Delta}} \log_e \dfrac{f\sqrt{X} - \sqrt{f\Delta}}{f\sqrt{X} + \sqrt{f\Delta}} & (f\Delta > 0) \end{cases}$

150. $\int \sqrt{XY}\, dx = \dfrac{\Delta + 2aY}{4af} \sqrt{XY} - \dfrac{\Delta^2}{8af} \int \dfrac{dx}{\sqrt{XY}}$

151. $\int \sqrt{\dfrac{Y}{X}}\, dx = \dfrac{1}{a}\sqrt{XY} - \dfrac{\Delta}{2a} \int \dfrac{dx}{\sqrt{XY}}$

152. $\int \dfrac{\sqrt{X}\, dx}{Y} = \dfrac{2\sqrt{X}}{f} + \dfrac{\Delta}{f} \int \dfrac{dx}{Y\sqrt{X}}$ $\qquad (f \neq 0)$

153. $\int \dfrac{Y^n\, dx}{\sqrt{X}} = \dfrac{2}{(2n+1)a}\left(\sqrt{X}Y^n - n\Delta \int \dfrac{Y^{n-1}\, dx}{\sqrt{X}}\right)$

154. $\int \dfrac{dx}{\sqrt{X}Y^n} = -\dfrac{1}{(n-1)\Delta}\left\{\dfrac{\sqrt{X}}{Y^{n-1}} + \left(n - \dfrac{3}{2}\right)a \int \dfrac{dx}{\sqrt{X}Y^{n-1}}\right\}$

$(\Delta \neq 0, n \neq 1)$

155. $\int \sqrt{X}Y^n\, dx = \dfrac{1}{(2n+3)f}\left(2\sqrt{X}Y^{n+1} + \Delta \int \dfrac{Y^n\, dx}{\sqrt{X}}\right)$ $\qquad (f \neq 0)$

156. $\int \dfrac{\sqrt{X}\, dx}{Y^n} = \dfrac{1}{(n-1)f}\left(-\dfrac{\sqrt{X}}{Y^{n-1}} + \dfrac{a}{2} \int \dfrac{dx}{\sqrt{X}Y^{n-1}}\right)$ $\qquad (f \neq 0, n \neq 1)$

(l) Integrals containing $\sqrt{a^2 - x^2}$, with

$$X \equiv a^2 - x^2 \qquad (a > 0)$$

157. $\int \sqrt{X}\, dx = \dfrac{1}{2}\left(x\sqrt{X} + a^2 \arcsin \dfrac{x}{a}\right)$

158. $\int x\sqrt{X}\, dx = -\dfrac{1}{3}\sqrt{X^3}$

159. $\int x^2\sqrt{X}\,dx = -\dfrac{x}{4}\sqrt{X^3} + \dfrac{a^2}{8}\left(x\sqrt{X} + a^2\arcsin\dfrac{x}{a}\right)$

160. $\int x^3\sqrt{X}\,dx = \dfrac{\sqrt{X^5}}{5} - a^2\,\dfrac{\sqrt{X^3}}{3}$

161. $\int \dfrac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\log_e\dfrac{a+\sqrt{X}}{x}$

162. $\int \dfrac{\sqrt{X}}{x^2}\,dx = -\dfrac{\sqrt{X}}{x} - \arcsin\dfrac{x}{a}$

163. $\int \dfrac{\sqrt{X}}{x^3}\,dx = -\dfrac{\sqrt{X}}{2x^2} + \dfrac{1}{2a}\log_e\dfrac{a+\sqrt{X}}{x}$

164. $\int \dfrac{dx}{\sqrt{X}} = \arcsin\dfrac{x}{a}$

165. $\int \dfrac{x\,dx}{\sqrt{X}} = -\sqrt{X}$

166. $\int \dfrac{x^2\,dx}{\sqrt{X}} = -\dfrac{x}{2}\sqrt{X} + \dfrac{a^2}{2}\arcsin\dfrac{x}{a}$

167. $\int \dfrac{x^3\,dx}{\sqrt{X}} = \dfrac{\sqrt{X^3}}{3} - a^2\sqrt{X}$

168. $\int \dfrac{dx}{x\sqrt{X}} = -\dfrac{1}{a}\log_e\dfrac{a+\sqrt{X}}{x}$

169. $\int \dfrac{dx}{x^2\sqrt{X}} = -\dfrac{\sqrt{X}}{a^2x}$

170. $\int \dfrac{dx}{x^3\sqrt{X}} = -\dfrac{\sqrt{X}}{2a^2x^2} - \dfrac{1}{2a^3}\log_e\dfrac{a+\sqrt{X}}{x}$

171. $\int \sqrt{X^3}\,dx = \dfrac{1}{4}\left(x\sqrt{X^3} + \dfrac{3a^2x}{2}\sqrt{X} + \dfrac{3a^4}{2}\arcsin\dfrac{x}{a}\right)$

172. $\int x\sqrt{X^3}\,dx = -\dfrac{1}{5}\sqrt{X^5}$

173. $\int x^2\sqrt{X^3}\,dx = -\dfrac{x\sqrt{X^5}}{6} + \dfrac{a^2x\sqrt{X^3}}{24} + \dfrac{a^4x\sqrt{X}}{16} + \dfrac{a^6}{16}\arcsin\dfrac{x}{a}$

174. $\displaystyle\int x^3\sqrt{X^3}\,dx = \frac{\sqrt{X^7}}{7} - \frac{a^2\sqrt{X^5}}{5}$

175. $\displaystyle\int \frac{\sqrt{X^3}}{x}\,dx = \frac{\sqrt{X^3}}{3} + a^2\sqrt{X} - a^3\log_e\frac{a+\sqrt{X}}{x}$

176. $\displaystyle\int \frac{\sqrt{X^3}}{x^2}\,dx = -\frac{\sqrt{X^3}}{x} - \frac{3}{2}x\sqrt{X} - \frac{3}{2}a^2\arcsin\frac{x}{a}$

177. $\displaystyle\int \frac{\sqrt{X^3}}{x^3}\,dx = -\frac{\sqrt{X^3}}{2x^2} - \frac{3\sqrt{X}}{2} + \frac{3a}{2}\log_e\frac{a+\sqrt{X}}{x}$

178. $\displaystyle\int \frac{dx}{\sqrt{X^3}} = \frac{x}{a^2\sqrt{X}}$

179. $\displaystyle\int \frac{x\,dx}{\sqrt{X^3}} = \frac{1}{\sqrt{X}}$

180. $\displaystyle\int \frac{x^2\,dx}{\sqrt{X^3}} = \frac{x}{\sqrt{X}} - \arcsin\frac{x}{a}$

181. $\displaystyle\int \frac{x^3\,dx}{\sqrt{X^3}} = \sqrt{X} + \frac{a^2}{\sqrt{X}}$

182. $\displaystyle\int \frac{dx}{x\sqrt{X^3}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\log_e\frac{a+\sqrt{X}}{x}$

183. $\displaystyle\int \frac{dx}{x^2\sqrt{X^3}} = \frac{1}{a^4}\left(-\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}}\right)$

184. $\displaystyle\int \frac{dx}{x^3\sqrt{X^3}} = -\frac{1}{2a^2x^2\sqrt{X}} + \frac{3}{2a^4\sqrt{X}} - \frac{3}{2a^5}\log_e\frac{a+\sqrt{X}}{x}$

(m) Integrals containing $\sqrt{x^2+a^2}$, with

$$X \equiv x^2 + a^2 \qquad (a > 0)$$

185. $\displaystyle\int \sqrt{X}\,dx = \frac{1}{2}\left(x\sqrt{X} + a^2\sinh^{-1}\frac{x}{a}\right)$

$$= \frac{1}{2}\left[x\sqrt{X} + a^2\log_e(x+\sqrt{X})\right] + C_1$$

186. $\int x\sqrt{X}\,dx = \frac{1}{3}\sqrt{X^3}$

187. $\int x^2\sqrt{X}\,dx = \frac{x}{4}\sqrt{X^3} - \frac{a^2}{8}\left(x\sqrt{X} + a^2\sinh^{-1}\frac{x}{a}\right)$

$$= \frac{x}{4}\sqrt{X^3} - \frac{a^2}{8}[x\sqrt{X} + a^2\log_e(x + \sqrt{X})] + C_1$$

188. $\int x^3\sqrt{X}\,dx = \frac{\sqrt{X^5}}{5} - \frac{a^2\sqrt{X^3}}{3}$

189. $\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\log_e\frac{a + \sqrt{X}}{x}$

190. $\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \sinh^{-1}\frac{x}{a}$

$$= -\frac{\sqrt{X}}{x} + \log_e(x + \sqrt{X}) + C_1$$

191. $\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} - \frac{1}{2a}\log_e\frac{a + \sqrt{X}}{x}$

192. $\int \frac{dx}{\sqrt{X}} = \sinh^{-1}\frac{x}{a} = \log_e(x + \sqrt{X}) + C_1$

193. $\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X}$

194. $\int \frac{x^2\,dx}{\sqrt{X}} = \frac{x}{2}\sqrt{X} - \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$

$$= \frac{x}{2}\sqrt{X} - \frac{a^2}{2}\log_e(x + \sqrt{X}) + C_1$$

195. $\int \frac{x^3\,dx}{\sqrt{X}} = \frac{\sqrt{X^3}}{3} - a^2\sqrt{X}$

196. $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a}\log_e\frac{a + \sqrt{X}}{x}$

197. $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2 x}$

198. $\displaystyle\int \frac{dx}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2a^2x^2} + \frac{1}{2a^3}\log_e\frac{a+\sqrt{X}}{x}$

199. $\displaystyle\int \sqrt{X^3}\,dx = \frac{1}{4}\left(x\sqrt{X^3} + \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\sinh^{-1}\frac{x}{a}\right)$

$\displaystyle\qquad\qquad\quad = \frac{1}{4}\left[x\sqrt{X^3} + \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\log_e(x+\sqrt{X})\right] + C_1$

200. $\displaystyle\int x\sqrt{X^3}\,dx = \frac{1}{5}\sqrt{X^5}$

201. $\displaystyle\int x^2\sqrt{X^3}\,dx = \frac{x\sqrt{X^5}}{6} - \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16} - \frac{a^6}{16}\sinh^{-1}\frac{x}{a}$

$\displaystyle\qquad\qquad\qquad = \frac{x\sqrt{X^5}}{6} - \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16}$

$\displaystyle\qquad\qquad\qquad\qquad\qquad\qquad - \frac{a^6}{16}\log_e(x+\sqrt{X}) + C_1$

202. $\displaystyle\int x^3\sqrt{X^3}\,dx = \frac{\sqrt{X^7}}{7} - \frac{a^2\sqrt{X^5}}{5}$

203. $\displaystyle\int \frac{\sqrt{X^3}}{x}\,dx = \frac{\sqrt{X^3}}{3} + a^2\sqrt{X} - a^3\log_e\frac{a+\sqrt{X}}{x}$

204. $\displaystyle\int \frac{\sqrt{X^3}}{x^2}\,dx = -\frac{\sqrt{X^3}}{x} + \frac{3}{2}x\sqrt{X} + \frac{3}{2}a^2\sinh^{-1}\frac{x}{a}$

$\displaystyle\qquad\qquad\quad = -\frac{\sqrt{X^3}}{x} + \frac{3}{2}x\sqrt{X} + \frac{3}{2}a^2\log_e(x+\sqrt{X}) + C_1$

205. $\displaystyle\int \frac{\sqrt{X^3}}{x^3}\,dx = -\frac{\sqrt{X^3}}{2x^2} + \frac{3}{2}\sqrt{X} - \frac{3}{2}a\log_e\left(\frac{a+\sqrt{X}}{x}\right)$

206. $\displaystyle\int \frac{dx}{\sqrt{X^3}} = \frac{x}{a^2\sqrt{X}}$

207. $\displaystyle\int \frac{x\,dx}{\sqrt{X^3}} = -\frac{1}{\sqrt{X}}$

208. $\displaystyle\int \frac{x^2\,dx}{\sqrt{X^3}} = -\frac{x}{\sqrt{X}} + \sinh^{-1}\frac{x}{a}$

$\displaystyle\qquad\qquad\quad = -\frac{x}{\sqrt{X}} + \log_e(x+\sqrt{X}) + C_1$

209. $\displaystyle \int \frac{x^3\,dx}{\sqrt{X^3}} = \sqrt{X} + \frac{a^2}{\sqrt{X}}$

210. $\displaystyle \int \frac{dx}{x\sqrt{X^3}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\log_e \frac{a+\sqrt{X}}{x}$

211. $\displaystyle \int \frac{dx}{x^2\sqrt{X^3}} = -\frac{1}{a^4}\left(\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}}\right)$

212. $\displaystyle \int \frac{dx}{x^3\sqrt{X^3}} = -\frac{1}{2a^2 x^2 \sqrt{X}} - \frac{3}{2a^4\sqrt{X}} + \frac{3}{2a^5}\log_e \frac{a+\sqrt{X}}{x}$

(n) Integrals containing $\sqrt{x^2 - a^2}$, with

$$X \equiv x^2 - a^2 \qquad (a > 0)$$

213. $\displaystyle \int \sqrt{X}\,dx = \frac{1}{2}\left(x\sqrt{X} - a^2 \cosh^{-1}\frac{x}{a}\right)$

$$= \frac{1}{2}[x\sqrt{X} - a^2 \log_e (x + \sqrt{X})] + C_1$$

214. $\displaystyle \int x\sqrt{X}\,dx = \frac{1}{3}\sqrt{X^3}$

215. $\displaystyle \int x^2\sqrt{X}\,dx = \frac{x}{4}\sqrt{X^3} + \frac{a^2}{8}\left(x\sqrt{X} - a^2 \cosh^{-1}\frac{x}{a}\right)$

$$= \frac{x}{4}\sqrt{X^3} + \frac{a^2}{8}[x\sqrt{X} - a^2 \log_e (x + \sqrt{X})] + C_1$$

216. $\displaystyle \int x^3\sqrt{X}\,dx = \frac{\sqrt{X^5}}{5} + \frac{a^2\sqrt{X^3}}{3}$

217. $\displaystyle \int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a \arccos\frac{a}{x}$

218. $\displaystyle \int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \cosh^{-1}\frac{x}{a}$

$$= -\frac{\sqrt{X}}{x} + \log_e (x + \sqrt{X}) + C_1$$

219. $\displaystyle\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a}\arccos\frac{a}{x}$

220. $\displaystyle\int \frac{dx}{\sqrt{X}} = \cosh^{-1}\frac{x}{a} = \log_e (x + \sqrt{X}) + C_1$

221. $\displaystyle\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X}$

222. $\displaystyle\int \frac{x^2\,dx}{\sqrt{X}} = \frac{x}{2}\sqrt{X} + \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$

$\displaystyle\qquad = \frac{x}{2}\sqrt{X} + \frac{a^2}{2}\log_e (x + \sqrt{X}) + C_1$

223. $\displaystyle\int \frac{x^3\,dx}{\sqrt{X}} = \frac{\sqrt{X^3}}{3} + a^2\sqrt{X}$

224. $\displaystyle\int \frac{dx}{x\sqrt{X}} = \frac{1}{a}\arccos\frac{a}{x}$

225. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = \frac{\sqrt{X}}{a^2 x}$

226. $\displaystyle\int \frac{dx}{x^3\sqrt{X}} = \frac{\sqrt{X}}{2a^2 x^2} + \frac{1}{2a^3}\arccos\frac{a}{x}$

227. $\displaystyle\int \sqrt{X^3}\,dx = \frac{1}{4}\left(x\sqrt{X^3} - \frac{3a^2 x}{2}\sqrt{X} + \frac{3a^4}{2}\cosh^{-1}\frac{x}{a}\right)$

$\displaystyle\qquad = \frac{1}{4}\left[x\sqrt{X^3} - \frac{3a^2 x}{2}\sqrt{X} + \frac{3a^4}{2}\log_e (x + \sqrt{X})\right] + C_1$

228. $\displaystyle\int x\sqrt{X^3}\,dx = \frac{1}{5}\sqrt{X^5}$

229. $\displaystyle\int x^2\sqrt{X^3}\,dx = \frac{x\sqrt{X^5}}{6} + \frac{a^2 x\sqrt{X^3}}{24} - \frac{a^4 x\sqrt{X}}{16} + \frac{a^6}{16}\cosh^{-1}\frac{x}{a}$

$\displaystyle\qquad = \frac{x\sqrt{X^5}}{6} + \frac{a^2 x\sqrt{X^3}}{24} - \frac{a^4 x\sqrt{X}}{16}$

$\displaystyle\qquad\qquad\qquad + \frac{a^6}{16}\log_e (x + \sqrt{X}) + C_1$

230. $\int x^3 \sqrt{X^3}\, dx \dfrac{\sqrt{X^7}}{7} + \dfrac{a^2\sqrt{X^5}}{5}$

231. $\int \dfrac{\sqrt{X^3}}{x}\, dx = \dfrac{\sqrt{X^3}}{3} - a^2\sqrt{X} + a^3 \arccos \dfrac{a}{x}$

232. $\int \dfrac{\sqrt{X^3}}{x^2}\, dx = -\dfrac{\sqrt{X^3}}{2} + \dfrac{3}{2}x\sqrt{X} - \dfrac{3}{2}a^2 \cosh^{-1} \dfrac{x}{a}$

$\qquad = -\dfrac{\sqrt{X^3}}{2} + \dfrac{3}{2}x\sqrt{X} - \dfrac{3}{2}a^2 \log_e (x + \sqrt{X}) + C_1$

233. $\int \dfrac{\sqrt{X^3}}{x^3}\, dx = -\dfrac{\sqrt{X^3}}{2x^2} + \dfrac{3\sqrt{X}}{2} - \dfrac{3}{2}a \arccos \dfrac{a}{x}$

234. $\int \dfrac{dx}{\sqrt{X^3}} = -\dfrac{x}{a^2\sqrt{X}}$

235. $\int \dfrac{x\, dx}{\sqrt{X^3}} = -\dfrac{1}{\sqrt{X}}$

236. $\int \dfrac{x^2\, dx}{\sqrt{X^3}} = -\dfrac{x}{\sqrt{X}} + \cosh^{-1} \dfrac{x}{a}$

$\qquad = -\dfrac{x}{\sqrt{X}} + \log_e (x + \sqrt{X}) + C_1$

237. $\int \dfrac{x^3\, dx}{\sqrt{X^3}} = \sqrt{X} - \dfrac{a^2}{\sqrt{X}}$

238. $\int \dfrac{dx}{x\sqrt{X^3}} = -\dfrac{1}{a^2\sqrt{X}} - \dfrac{1}{a^3} \arccos \dfrac{a}{x}$

239. $\int \dfrac{dx}{x^2\sqrt{X^3}} = -\dfrac{1}{a^4}\left(\dfrac{\sqrt{X}}{x} + \dfrac{x}{\sqrt{X}} \right)$

240. $\int \dfrac{dx}{x^3\sqrt{X^3}} = \dfrac{1}{2a^2x^2\sqrt{X}} - \dfrac{3}{2a^4\sqrt{X}} - \dfrac{3}{2a^5} \arccos \dfrac{a}{x}$

(o) Integrals containing $\sqrt{ax^2 + bx + c}$, with

$$X \equiv ax^2 + bx + c$$

$$\Delta = 4ac - b^2 \qquad k = \dfrac{4a}{4ac - b^2} \qquad\qquad (a \neq 0)$$

241. $\displaystyle\int \frac{dx}{\sqrt{X}} = \begin{cases} \dfrac{1}{\sqrt{a}}\log_e\left(2\sqrt{aX}+2ax+b\right)+C & (a>0) \\[2.5ex] \dfrac{1}{\sqrt{a}}\sinh^{-1}\dfrac{2ax+b}{\sqrt{\Delta}}+C_1 & (a>0,\ \Delta>0) \\[2.5ex] \dfrac{1}{\sqrt{a}}\log_e(2ax+b) & (a>0,\ \Delta=0) \\[2.5ex] -\dfrac{1}{\sqrt{-a}}\arcsin\dfrac{2ax+b}{\sqrt{-\Delta}} & (a<0,\ \Delta<0) \end{cases}$

242. $\displaystyle\int \frac{dx}{X\sqrt{X}} = \frac{2(2ax+b)}{\Delta\sqrt{X}} \qquad (\Delta \neq 0)$

243. $\displaystyle\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2ax+b)}{3\Delta\sqrt{X}}\left(\frac{1}{X}+2k\right) \qquad (\Delta \neq 0)$

244. $\displaystyle\int \frac{dx}{X^{(2n+1)/2}} = \frac{2(2ax+b)}{(2n-1)\Delta X^{(2n-1)/2}} + \frac{2k(n-1)}{2n-1}\int \frac{dx}{X^{(2n-1)/2}}$
$$(\Delta \neq 0)$$

245. $\displaystyle\int \sqrt{X}\,dx = \frac{(2ax+b)\sqrt{X}}{4a} + \frac{1}{2k}\int \frac{dx}{\sqrt{X}}$

246. $\displaystyle\int X\sqrt{X}\,dx = \frac{(2ax+b)\sqrt{X}}{8a}\left(X+\frac{3}{2k}\right) + \frac{3}{8k^2}\int \frac{dx}{\sqrt{X}}$

247. $\displaystyle\int X^2\sqrt{X}\,dx = \frac{(2ax+b)\sqrt{X}}{12a}\left(X^2+\frac{5X}{4k}+\frac{15}{8k^2}\right) + \frac{5}{16k^3}\int \frac{dx}{\sqrt{X}}$

248. $\displaystyle\int X^{(2n+1)/2}\,dx = \frac{(2ax+b)X^{(2n+1)/2}}{4a(n+1)} + \frac{2n+1}{2k(n+1)}\int X^{(2n-1)/2}\,dx$

249. $\displaystyle\int \frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{a} - \frac{b}{2a}\int \frac{dx}{\sqrt{X}}$

250. $\displaystyle\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2c)}{\Delta\sqrt{X}} \qquad (\Delta \neq 0)$

251. $\displaystyle\int \frac{x\,dx}{X^{(2n+1)/2}} = -\frac{1}{(2n-1)aX^{(2n-1)/2}} - \frac{b}{2a}\int \frac{dx}{X^{(2n+1)/2}}$

252. $\int \dfrac{x^2\,dx}{\sqrt{X}} = \left(\dfrac{x}{2a} - \dfrac{3b}{4a^2}\right)\sqrt{X} + \dfrac{3b^2 - 4ac}{8a^2}\int \dfrac{dx}{\sqrt{X}}$

253. $\int \dfrac{x^2\,dx}{X\sqrt{X}} = \dfrac{(2b^2 - 4ac)x + 2bc}{a\Delta\sqrt{X}} + \dfrac{1}{a}\int \dfrac{dx}{\sqrt{X}}$ $\quad (\Delta \neq 0)$

254. $\int x\sqrt{X}\,dx = \dfrac{X\sqrt{X}}{3a} - \dfrac{b(2ax + b)}{8a^2}\sqrt{X} - \dfrac{b}{4ak}\int \dfrac{dx}{\sqrt{X}}$

255. $\int xX\sqrt{X}\,dx = \dfrac{X^2\sqrt{X}}{5a} - \dfrac{b}{2a}\int X\sqrt{X}\,dx$

256. $\int xX^{(2n+1)/2}\,dx = \dfrac{X^{(2n+3)/2}}{(2n+3)a} - \dfrac{b}{2a}\int X^{(2n+1)/2}\,dx$

257. $\int x^2\sqrt{X}\,dx = \left(x - \dfrac{5b}{6a}\right)\dfrac{X\sqrt{X}}{4a} + \dfrac{5b^2 - 4ac}{16a^2}\int \sqrt{X}\,dx$

258. $\int \dfrac{dx}{x\sqrt{X}} = \begin{cases} -\dfrac{1}{\sqrt{c}}\log_e\left(\dfrac{2\sqrt{cX}}{x} + \dfrac{2c}{x} + b\right) + C & (c > 0) \\[2ex] -\dfrac{1}{\sqrt{c}}\sinh^{-1}\dfrac{bx + 2c}{x\sqrt{\Delta}} + C_1 & (c > 0, \Delta > 0) \\[2ex] -\dfrac{1}{\sqrt{c}}\log_e\dfrac{bx + 2c}{x} & (c > 0, \Delta = 0) \\[2ex] \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx + 2c}{x\sqrt{-\Delta}} & (c < 0, \Delta < 0) \end{cases}$

259. $\int \dfrac{dx}{x^2\sqrt{X}} = -\dfrac{\sqrt{X}}{cx} - \dfrac{b}{2c}\int \dfrac{dx}{x\sqrt{X}}$

260. $\int \dfrac{\sqrt{X}\,dx}{x} = \sqrt{X} + \dfrac{b}{2}\int \dfrac{dx}{\sqrt{X}} + c\int \dfrac{dx}{x\sqrt{X}}$

261. $\int \dfrac{\sqrt{X}\,dx}{x^2} = -\dfrac{\sqrt{X}}{x} + a\int \dfrac{dx}{\sqrt{X}} + \dfrac{b}{2}\int \dfrac{dx}{x\sqrt{X}}$

262. $\int \dfrac{X^{(2n+1)/2}}{x}\,dx = \dfrac{X^{(2n+1)/2}}{2n+1} + \dfrac{b}{2}\int X^{(2n-1)/2}\,dx + c\int \dfrac{X^{(2n-1)/2}}{x}\,dx$

(p) *Other irrational forms* $\qquad (a > 0, b \neq 0)$

263. $\displaystyle \int \frac{dx}{x\sqrt{ax^2 + bx}} = -\frac{2}{bx}\sqrt{ax^2 + bx}$

264. $\displaystyle \int \frac{dx}{\sqrt{2ax - x^2}} = \arcsin \frac{x - a}{a}$

265. $\displaystyle \int \frac{x\,dx}{\sqrt{2ax - x^2}} = -\sqrt{2ax - x^2} + a \arcsin \frac{x - a}{a}$

266. $\displaystyle \int \sqrt{2ax - x^2}\,dx = \frac{x - a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\arcsin \frac{x - a}{a}$

267. $\displaystyle \int \frac{dx}{(ax^2 + b)\sqrt{fx^2 + g}} = \frac{1}{\sqrt{b}\,\sqrt{ag - bf}}\arctan \frac{x\sqrt{ag - bf}}{\sqrt{b}\,\sqrt{fx^2 + g}}$

$$(ag - bf > 0)$$

$$= \frac{1}{2\sqrt{b}\,\sqrt{bf - ag}}\log_e \frac{\sqrt{b}\,\sqrt{fx^2 + g} + x\sqrt{bf - ag}}{\sqrt{b}\,\sqrt{fx^2 + g} - x\sqrt{bf - ag}} \quad (ag - bf < 0)$$

268. $\displaystyle \int \sqrt[n]{ax + b}\,dx = \frac{n(ax + b)}{(n + 1)a}\sqrt[n]{ax + b}$

269. $\displaystyle \int \frac{dx}{\sqrt[n]{ax + b}}\,dx = \frac{n(ax + b)}{(n - 1)a}\frac{1}{\sqrt[n]{ax + b}}$

270. $\displaystyle \int \frac{dx}{x\sqrt{x^n + a^2}} = -\frac{2}{na}\log_e \frac{a + \sqrt{x^n + a^2}}{\sqrt{x^n}}$

271. $\displaystyle \int \frac{dx}{x\sqrt{x^n - a^2}} = \frac{2}{na}\arccos \frac{a}{\sqrt{x^n}}$

272. $\displaystyle \int \frac{\sqrt{x}\,dx}{\sqrt{a^3 - x^3}} = \frac{2}{3}\arcsin \sqrt{\left(\frac{x}{a}\right)^3}$

(q) *Recursion formulas* $(m, n, p$ *are integers)*

273. $\displaystyle \int x^m(ax^n + b)^p\,dx$

$$= \frac{1}{m + np + 1}\left[x^{m+1}(ax^n + b)^p + npb \int x^m(ax^n + b)^{p-1}\,dx\right]$$

$$= \frac{1}{bn(p+1)}\left[-x^{m+1}(ax^n+b)^{p+1}\right.$$
$$\left.+ (m+n+np+1)\int x^m(ax^n+b)^{p+1}\,dx\right]$$

$$= \frac{1}{(m+1)b}\left[x^{m+1}(ax^n+b)^{p+1}\right.$$
$$\left.- a(m+n+np+1)\int x^{m+n}(ax^n+b)^p\,dx\right]$$

$$= \frac{1}{a(m+np+1)}\left[x^{m-n+1}(ax^n+b)^{p+1}\right.$$
$$\left.- (m-n+1)b\int x^{m-n}(ax^n+b)^p\,dx.\right]$$

(r) Integrals containing the sine function   ($a \neq 0$)

274. $\displaystyle\int \sin ax\,dx = -\frac{1}{a}\cos ax$

275. $\displaystyle\int \sin^2 ax\,dx = \frac{1}{2}x - \frac{1}{4a}\sin 2ax$

276. $\displaystyle\int \sin^3 ax\,dx = -\frac{1}{a}\cos ax + \frac{1}{3a}\cos^3 ax$

277. $\displaystyle\int \sin^4 ax\,dx = \frac{3}{8}x - \frac{1}{4a}\sin 2ax + \frac{1}{32a}\sin 4ax$

278. $\displaystyle\int \sin^n ax\,dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax\,dx$ ($n > 0$)

279. $\displaystyle\int x \sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a}$

280. $\displaystyle\int x^2 \sin ax\,dx = \frac{2x}{a^2}\sin ax - \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\cos ax$

281. $\displaystyle\int x^3 \sin ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\sin ax - \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\cos ax$

282. $\displaystyle\int x^n \sin ax\,dx = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx$

283. $\displaystyle\int \frac{\sin ax}{x}\,dx = ax - \frac{(ax)^3}{3 \cdot 3!} + \frac{(ax)^5}{5 \cdot 5!} - \frac{(ax)^7}{7 \cdot 7!} + \cdots$

284. $\displaystyle\int \frac{\sin ax}{x^2}\, dx = -\frac{\sin ax}{x} + a \int \frac{\cos ax\, dx}{x}$

285. $\displaystyle\int \frac{\sin ax}{x^n}\, dx = -\frac{1}{n-1}\frac{\sin ax}{x^{n-1}} + \frac{a}{n-1} \int \frac{\cos ax}{x^{n-1}}\, dx$

286. $\displaystyle\int \frac{dx}{\sin ax} = \int \operatorname{cosec} ax\, dx = \frac{1}{a} \log_e \tan \frac{ax}{2}$

$$= \frac{1}{a} \log_e (\operatorname{cosec} ax - \cot ax)$$

287. $\displaystyle\int \frac{dx}{\sin^2 ax} = -\frac{1}{a} \cot ax$

288. $\displaystyle\int \frac{dx}{\sin^3 ax} = -\frac{\cos ax}{2a \sin^2 ax} + \frac{1}{2a} \log_e \tan \frac{ax}{2}$

289. $\displaystyle\int \frac{dx}{\sin^n ax} = -\frac{1}{a(n-1)}\frac{\cos ax}{\sin^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} ax}$ $(n > 1)$

290. $\displaystyle\int \frac{x\, dx}{\sin ax} = \frac{1}{a^2}\left[ax + \frac{(ax)^3}{3 \cdot 3!} + \frac{7(ax)^5}{3 \cdot 5 \cdot 5!} + \frac{31(ax)^7}{3 \cdot 7 \cdot 7!}\right.$

$$\left. + \frac{127(ax)^9}{3 \cdot 5 \cdot 9!} + \cdots \right]$$

291. $\displaystyle\int \frac{x\, dx}{\sin^2 ax} = -\frac{x}{a} \cot ax + \frac{1}{a^2} \log_e \sin ax$

292. $\displaystyle\int \frac{x\, dx}{\sin^n ax} = -\frac{x \cos ax}{(n-1)a \sin^{n-1} ax} - \frac{1}{(n-1)(n-2)a^2 \sin^{n-2} ax}$

$$+ \frac{n-2}{n-1} \int \frac{x\, dx}{\sin^{n-2} ax} \qquad (n > 2)$$

293. $\displaystyle\int \frac{dx}{1 + \sin ax} = -\frac{1}{a} \tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

294. $\displaystyle\int \frac{dx}{1 - \sin ax} = \frac{1}{a} \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

295. $\displaystyle\int \frac{x\, dx}{1 + \sin ax} = -\frac{x}{a} \tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2} \log_e \cos\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

296. $\displaystyle\int \frac{x\, dx}{1 - \sin ax} = \frac{x}{a} \cot\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2} \log_e \sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)$

297. $\int \dfrac{\sin ax \, dx}{1 \pm \sin ax} = \pm x + \dfrac{1}{a} \tan \left(\dfrac{\pi}{4} \mp \dfrac{ax}{2} \right)$

298. $\int \dfrac{dx}{\sin ax \, (1 \pm \sin ax)} = \dfrac{1}{a} \tan \left(\dfrac{\pi}{4} \mp \dfrac{ax}{2} \right) + \dfrac{1}{a} \log_e \tan \dfrac{ax}{2}$

299. $\int \dfrac{dx}{(1 + \sin ax)^2} = -\dfrac{1}{2a} \tan \left(\dfrac{\pi}{4} - \dfrac{ax}{2} \right) - \dfrac{1}{6a} \tan^3 \left(\dfrac{\pi}{4} - \dfrac{ax}{2} \right)$

300. $\int \dfrac{dx}{(1 - \sin ax)^2} = \dfrac{1}{2a} \cot \left(\dfrac{\pi}{4} - \dfrac{ax}{2} \right) + \dfrac{1}{6a} \cot^3 \left(\dfrac{\pi}{4} - \dfrac{ax}{2} \right)$

301. $\int \dfrac{\sin ax \, dx}{(1 + \sin ax)^2} = -\dfrac{1}{2a} \tan \left(\dfrac{\pi}{4} - \dfrac{ax}{2} \right) + \dfrac{1}{6a} \tan^3 \left(\dfrac{\pi}{4} - \dfrac{ax}{2} \right)$

302. $\int \dfrac{\sin ax \, dx}{(1 - \sin ax)^2} = -\dfrac{1}{2a} \cot \left(\dfrac{\pi}{4} - \dfrac{ax}{2} \right) + \dfrac{1}{6a} \cot^3 \left(\dfrac{\pi}{4} - \dfrac{ax}{2} \right)$

303. $\int \dfrac{dx}{1 + \sin^2 ax} = \dfrac{1}{2\sqrt{2}a} \arcsin \left(\dfrac{3 \sin^2 ax - 1}{\sin^2 ax + 1} \right)$

304. $\int \dfrac{dx}{1 - \sin^2 ax} = \int \dfrac{dx}{\cos^2 ax} = \dfrac{1}{a} \tan ax$

305. $\int \sin ax \sin bx \, dx = \dfrac{\sin (a - b)x}{2(a - b)} - \dfrac{\sin (a + b)x}{2(a + b)}$

$$(|a| \neq |b|; \text{ for } |a| = |b| \text{ see } 275)$$

306. $\int \dfrac{dx}{b + c \sin ax} = \dfrac{2}{a\sqrt{b^2 - c^2}} \arctan \dfrac{b \tan (ax/2) + c}{\sqrt{b^2 - c^2}} \qquad (b^2 > c^2)$

$$= \dfrac{1}{a\sqrt{c^2 - b^2}} \log_e \dfrac{b \tan (ax/2) + c - \sqrt{c^2 - b^2}}{b \tan (ax/2) + c + \sqrt{c^2 - b^2}}$$

$$(b^2 < c^2)$$

307. $\int \dfrac{\sin ax \, dx}{b + c \sin ax} = \dfrac{x}{c} - \dfrac{b}{c} \int \dfrac{dx}{b + c \sin ax}$

308. $\int \dfrac{dx}{\sin ax \, (b + c \sin ax)} = \dfrac{1}{ab} \log_e \tan \dfrac{ax}{2} - \dfrac{c}{b} \int \dfrac{dx}{b + c \sin ax}$

309. $\int \dfrac{dx}{(b + c \sin ax)^2} = \dfrac{c \cos ax}{a(b^2 - c^2)(b + c \sin ax)}$

$$+ \dfrac{b}{b^2 - c^2} \int \dfrac{dx}{b + c \sin ax}$$

310. $\displaystyle\int \frac{\sin ax\, dx}{(b + c \sin ax)^2} = \frac{b \cos ax}{a(c^2 - b^2)(b + c \sin ax)}$

$$+ \frac{c}{c^2 - b^2} \int \frac{dx}{b + c \sin ax}$$

311. $\displaystyle\int \frac{dx}{b^2 + c^2 \sin^2 ax} = \frac{1}{ab\sqrt{b^2 + c^2}} \arctan \frac{\sqrt{b^2 + c^2}\, \tan ax}{b} \quad (b > 0)$

312. $\displaystyle\int \frac{dx}{b^2 - c^2 \sin^2 ax} = \frac{1}{ab\sqrt{b^2 - c^2}} \arctan \frac{\sqrt{b^2 - c^2}\, \tan ax}{b}$

$$(b^2 > c^2,\ b > 0)$$

$$= \frac{1}{2ab\sqrt{c^2 - b^2}} \log_e \frac{\sqrt{c^2 - b^2}\, \tan ax + b}{\sqrt{c^2 - b^2}\, \tan ax - b}$$

$$(c^2 > b^2,\ b > 0)$$

(s) Integrals containing the cosine function $(a \neq 0)$

313. $\displaystyle\int \cos ax\, dx = \frac{1}{a} \sin ax$

314. $\displaystyle\int \cos^2 ax\, dx = \frac{1}{2}x + \frac{1}{4a} \sin 2ax$

315. $\displaystyle\int \cos^3 ax\, dx = \frac{1}{a} \sin ax - \frac{1}{3a} \sin^3 ax$

316. $\displaystyle\int \cos^4 ax\, dx = \frac{3}{8}x + \frac{1}{4a} \sin 2ax + \frac{1}{32a} \sin 4ax$

317. $\displaystyle\int \cos^n ax\, dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n} \int \cos^{n-2} ax\, dx$

318. $\displaystyle\int x \cos ax\, dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a}$

319. $\displaystyle\int x^2 \cos ax\, dx = \frac{2x}{a^2} \cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right) \sin ax$

320. $\displaystyle\int x^3 \cos ax\, dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right) \cos ax + \left(\frac{x^3}{a} - \frac{6x}{a^3}\right) \sin ax$

321. $\displaystyle\int x^n \cos ax\, dx = \frac{x^n \sin ax}{a} - \frac{n}{a} \int x^{n-1} \sin ax\, dx \qquad (n > 0)$

322. $\displaystyle\int \frac{\cos ax}{x}\,dx = \log_e (ax) - \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} - \frac{(ax)^6}{6\cdot 6!} + \cdots$

323. $\displaystyle\int \frac{\cos ax}{x^2}\,dx = -\frac{\cos ax}{x} - a\int \frac{\sin ax\,dx}{x}$

324. $\displaystyle\int \frac{\cos ax}{x^n}\,dx = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1}\int \frac{\sin ax\,dx}{x^{n-1}}$ $\qquad(n \neq 1)$

325. $\displaystyle\int \frac{dx}{\cos ax} = \frac{1}{a}\log_e \tan\left(\frac{ax}{2} + \frac{\pi}{4}\right) = \frac{1}{a}\log_e (\sec ax + \tan ax)$

326. $\displaystyle\int \frac{dx}{\cos^2 ax} = \frac{1}{a}\tan ax$

327. $\displaystyle\int \frac{dx}{\cos^3 ax} = \frac{\sin ax}{2a\cos^2 ax} + \frac{1}{2a}\log_e \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

328. $\displaystyle\int \frac{dx}{\cos^n ax} = \frac{1}{a(n-1)}\frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} ax}$ $\qquad(n > 1)$

329. $\displaystyle\int \frac{x\,dx}{\cos ax} = \frac{1}{a^2}\left[\frac{(ax)^2}{2} + \frac{(ax)^4}{4\cdot 2!} + \frac{5(ax)^6}{6\cdot 4!} + \frac{61(ax)^8}{8\cdot 6!} \right.$
$$\left. + \frac{1{,}385(ax)^{10}}{10\cdot 8!} + \cdots\right]$$

330. $\displaystyle\int \frac{x\,dx}{\cos^2 ax} = \frac{x}{a}\tan ax + \frac{1}{a^2}\log_e \cos ax$

331. $\displaystyle\int \frac{x\,dx}{\cos^n ax} = \frac{x\sin ax}{(n-1)a\cos^{n-1} ax} - \frac{1}{(n-1)(n-2)a^2\cos^{n-2} ax}$
$$+ \frac{n-2}{n-1}\int \frac{x\,dx}{\cos^{n-2} ax} \qquad(n > 2)$$

332. $\displaystyle\int \frac{dx}{1 + \cos ax} = \frac{1}{a}\tan \frac{ax}{2}$

333. $\displaystyle\int \frac{dx}{1 - \cos ax} = -\frac{1}{a}\cot \frac{ax}{2}$

334. $\displaystyle\int \frac{x\,dx}{1 + \cos ax} = \frac{x}{a}\tan \frac{ax}{2} + \frac{2}{a^2}\log_e \cos \frac{ax}{2}$

335. $\displaystyle\int \frac{x\,dx}{1 - \cos ax} = -\frac{x}{a}\cot \frac{ax}{2} + \frac{2}{a^2}\log_e \sin \frac{ax}{2}$

336. $\displaystyle\int \frac{\cos ax\, dx}{1 + \cos ax} = x - \frac{1}{a}\tan\frac{ax}{2}$

337. $\displaystyle\int \frac{\cos ax\, dx}{1 - \cos ax} = -x - \frac{1}{a}\cot\frac{ax}{2}$

338. $\displaystyle\int \frac{dx}{\cos ax\,(1 + \cos ax)} = \frac{1}{a}\log_e\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) - \frac{1}{a}\tan\frac{ax}{2}$

339. $\displaystyle\int \frac{dx}{\cos ax\,(1 - \cos ax)} = \frac{1}{a}\log_e\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) - \frac{1}{a}\cot\frac{ax}{2}$

340. $\displaystyle\int \frac{dx}{(1 + \cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^3\frac{ax}{2}$

341. $\displaystyle\int \frac{dx}{(1 - \cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$

342. $\displaystyle\int \frac{\cos ax\, dx}{(1 + \cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} - \frac{1}{6a}\tan^3\frac{ax}{2}$

343. $\displaystyle\int \frac{\cos ax\, dx}{(1 - \cos ax)^2} = \frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$

344. $\displaystyle\int \frac{dx}{1 + \cos^2 ax} = \frac{1}{2\sqrt{2}\,a}\arcsin\left(\frac{1 - 3\cos^2 ax}{1 + \cos^2 ax}\right)$

345. $\displaystyle\int \frac{dx}{1 - \cos^2 ax} = \int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax$

346. $\displaystyle\int \cos ax \cos bx\, dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}$

$(|a| \neq |b|;\ \text{for}\ |a| = |b|,\ \text{see } 314)$

347. $\displaystyle\int \frac{dx}{b + c\cos ax} = \frac{2}{a\sqrt{b^2 - c^2}}\arctan\frac{(b-c)\tan(ax/2)}{\sqrt{b^2 - c^2}}\qquad (b^2 > c^2)$

$\displaystyle\qquad\qquad = \frac{1}{a\sqrt{c^2 - b^2}}\log_e\frac{(c-b)\tan(ax/2) + \sqrt{c^2 - b^2}}{(c-b)\tan(ax/2) - \sqrt{c^2 - b^2}}$

$(b^2 < c^2)$

348. $\displaystyle\int \frac{\cos ax\, dx}{b + c\cos ax} = \frac{x}{c} - \frac{b}{c}\int \frac{dx}{b + c\cos ax}$

349. $\int \dfrac{dx}{\cos ax\,(b + c \cos ax)} = \dfrac{1}{ab} \log_e \tan \left(\dfrac{ax}{2} + \dfrac{\pi}{4} \right) - \dfrac{c}{b} \int \dfrac{dx}{b + c \cos ax}$

350. $\int \dfrac{dx}{(b + c \cos ax)^2} = \dfrac{c \sin ax}{a(c^2 - b^2)(b + c \cos ax)}$

$$- \dfrac{b}{c^2 - b^2} \int \dfrac{dx}{b + c \cos ax}$$

351. $\int \dfrac{\cos ax\,dx}{(b + c \cos ax)^2} = \dfrac{b \sin ax}{a(b^2 - c^2)(b + c \cos ax)}$

$$- \dfrac{c}{b^2 - c^2} \int \dfrac{dx}{b + c \cos ax}$$

352. $\int \dfrac{dx}{b^2 + c^2 \cos^2 ax} = \dfrac{1}{ab\sqrt{b^2 + c^2}} \arctan \dfrac{b \tan ax}{\sqrt{b^2 + c^2}}$ $\qquad (b > 0)$

353. $\int \dfrac{dx}{b^2 - c^2 \cos^2 ax} = \dfrac{1}{ab\sqrt{b^2 - c^2}} \arctan \dfrac{b \tan ax}{\sqrt{b^2 - c^2}}$ $\quad (b^2 > c^2, b > 0)$

$$= \dfrac{1}{2ab\sqrt{c^2 - b^2}} \log_e \dfrac{b \tan ax - \sqrt{c^2 - b^2}}{b \tan ax + \sqrt{c^2 - b^2}}$$

$$(c^2 > b^2, b > 0)$$

(t) Integrals containing both sine and cosine $\qquad (a \neq 0)$

354. $\int \sin ax \cos ax\,dx = \dfrac{1}{2a} \sin^2 ax$

355. $\int \sin^2 ax \cos^2 ax\,dx = \dfrac{x}{8} - \dfrac{\sin 4ax}{32a}$

356. $\int \sin^n ax \cos ax\,dx = \dfrac{1}{a(n + 1)} \sin^{n+1} ax$ $\qquad (n \neq -1)$

357. $\int \sin ax \cos^n ax\,dx = -\dfrac{1}{a(n + 1)} \cos^{n+1} ax$ $\qquad (n \neq -1)$

358. $\int \sin^n ax \cos^m ax\,dx$

$$= -\dfrac{\sin^{n-1} ax \cos^{m+1} ax}{a(n + m)} + \dfrac{n - 1}{n + m} \int \sin^{n-2} ax \cos^m ax\,dx$$

$$(m,\ n > 0)$$

$$= \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(n + m)} + \frac{m - 1}{n + m} \int \sin^n ax \cos^{m-2} ax \, dx$$

$$(m, n > 0)$$

359. $\int \frac{dx}{\sin ax \cos ax} = \frac{1}{a} \log_e \tan ax$

360. $\int \frac{dx}{\sin^2 ax \cos ax} = \frac{1}{a} \left[\log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) - \frac{1}{\sin ax} \right]$

361. $\int \frac{dx}{\sin ax \cos^2 ax} = \frac{1}{a} \left(\log_e \tan \frac{ax}{2} + \frac{1}{\cos ax} \right)$

362. $\int \frac{dx}{\sin^3 ax \cos ax} = \frac{1}{a} \left(\log_e \tan ax - \frac{1}{2 \sin^2 ax} \right)$

363. $\int \frac{dx}{\sin ax \cos^3 ax} = \frac{1}{a} \left(\log_e \tan ax + \frac{1}{2 \cos^2 ax} \right)$

364. $\int \frac{dx}{\sin^2 ax \cos^2 ax} = -\frac{2}{a} \cot 2ax$

365. $\int \frac{dx}{\sin^2 ax \cos^3 ax} = \frac{1}{a} \left[\frac{\sin ax}{2 \cos^2 ax} - \frac{1}{\sin ax} + \frac{3}{2} \log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) \right]$

366. $\int \frac{dx}{\sin^3 ax \cos^2 ax} = \frac{1}{a} \left(\frac{1}{\cos ax} - \frac{\cos ax}{2 \sin^2 ax} + \frac{3}{2} \log_e \tan \frac{ax}{2} \right)$

367. $\int \frac{dx}{\sin ax \cos^n ax} = \frac{1}{a(n - 1) \cos^{n-1} ax} + \int \frac{dx}{\sin ax \cos^{n-2} ax}$

$$(n \neq 1); \text{ (see 361, 363)}$$

368. $\int \frac{dx}{\sin^n ax \cos ax} = -\frac{1}{a(n - 1) \sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} ax \cos ax}$

$$(n \neq 1); \text{ (see 360, 362)}$$

369. $\int \frac{dx}{\sin^n ax \cos^m ax}$

$$= -\frac{1}{a(n - 1)} \cdot \frac{1}{\sin^{n-1} ax \cos^{m-1} ax} + \frac{n + m - 2}{n - 1} \int \frac{dx}{\sin^{n-2} ax \cos^m ax}$$

$$(m > 0, n > 1)$$

$$= \frac{1}{a(m - 1)} \cdot \frac{1}{\sin^{n-1} ax \cos^{m-1} ax} + \frac{n + m - 2}{m - 1} \int \frac{dx}{\sin^n ax \cos^{m-2} ax}$$

$$(n > 0, m > 1)$$

370. $\displaystyle\int \frac{\sin ax\, dx}{\cos^2 ax} = \frac{1}{a \cos ax}$

371. $\displaystyle\int \frac{\sin ax\, dx}{\cos^3 ax} = \frac{1}{2a \cos^2 ax} = \frac{1}{2a} \tan^2 ax + C_1$

372. $\displaystyle\int \frac{\sin ax\, dx}{\cos^n ax} = \frac{1}{a(n-1) \cos^{n-1} ax}$

373. $\displaystyle\int \frac{\sin^2 ax\, dx}{\cos ax} = -\frac{1}{a} \sin ax + \frac{1}{a} \log_e \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

374. $\displaystyle\int \frac{\sin^2 ax\, dx}{\cos^3 ax} = \frac{1}{a}\left[\frac{\sin ax}{2 \cos^2 ax} - \frac{1}{2} \log_e \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right]$

375. $\displaystyle\int \frac{\sin^2 ax\, dx}{\cos^n ax} = \frac{\sin ax}{a(n-1) \cos^{n-1} ax} - \frac{1}{n-1} \int \frac{dx}{\cos^{n-2} ax}$
$$(n \neq 1) \qquad \text{(see 325, 326, 328)}$$

376. $\displaystyle\int \frac{\sin^3 ax\, dx}{\cos ax} = -\frac{1}{a}\left(\frac{\sin^2 ax}{2} + \log_e \cos ax\right)$

377. $\displaystyle\int \frac{\sin^3 ax\, dx}{\cos^2 ax} = \frac{1}{a}\left(\cos ax + \frac{1}{\cos ax}\right)$

378. $\displaystyle\int \frac{\sin^3 ax\, dx}{\cos^n ax} = \frac{1}{a}\left[\frac{1}{(n-1) \cos^{n-1} ax} - \frac{1}{(n-3) \cos^{n-3} ax}\right]$
$$(n \neq 1, n \neq 3)$$

379. $\displaystyle\int \frac{\sin^n ax}{\cos ax}\, dx = -\frac{\sin^{n-1} ax}{a(n-1)} + \int \frac{\sin^{n-2} ax\, dx}{\cos ax} \qquad (n \neq 1)$

380. $\displaystyle\int \frac{\sin^n ax}{\cos^m ax}\, dx = \frac{\sin^{n+1} ax}{a(m-1) \cos^{m-1} ax} - \frac{n-m+2}{m-1} \int \frac{\sin^n ax}{\cos^{m-2} ax}\, dx$
$$(m \neq 1)$$

$\displaystyle = -\frac{\sin^{n-1} ax}{a(n-m) \cos^{m-1} ax} + \frac{n-1}{n-m} \int \frac{\sin^{n-2} ax\, dx}{\cos^m ax}$
$$(m \neq n)$$

$\displaystyle = \frac{\sin^{n-1} ax}{a(m-1) \cos^{m-1} ax} - \frac{n-1}{m-1} \int \frac{\sin^{n-1} ax\, dx}{\cos^{m-2} ax}$
$$(m \neq 1)$$

381. $\displaystyle\int \frac{\cos ax\,dx}{\sin^2 ax} = -\frac{1}{a\sin ax}$

382. $\displaystyle\int \frac{\cos ax\,dx}{\sin^3 ax} = -\frac{1}{2a\sin^2 ax}$

383. $\displaystyle\int \frac{\cos ax\,dx}{\sin^n ax} = -\frac{1}{a(n-1)\sin^{n-1} ax} \qquad (n \neq 1)$

384. $\displaystyle\int \frac{\cos^2 ax\,dx}{\sin ax} = \frac{1}{a}\left(\cos ax + \log_e \tan \frac{ax}{2}\right)$

385. $\displaystyle\int \frac{\cos^2 ax\,dx}{\sin^3 ax} = -\frac{1}{2a}\left(\frac{\cos ax}{\sin^2 ax} - \log_e \tan \frac{ax}{2}\right)$

386. $\displaystyle\int \frac{\cos^2 ax\,dx}{\sin^n ax} = -\frac{1}{n-1}\left(\frac{\cos ax}{a\sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} ax}\right)$

$$(n \neq 1); \text{ (see 289)}$$

387. $\displaystyle\int \frac{\cos^3 ax\,dx}{\sin ax} = \frac{1}{a}\left(\frac{\cos^2 ax}{2} + \log_e \sin ax\right)$

388. $\displaystyle\int \frac{\cos^3 ax\,dx}{\sin^2 ax} = -\frac{1}{a}\left(\sin ax + \frac{1}{\sin ax}\right)$

389. $\displaystyle\int \frac{\cos^3 ax\,dx}{\sin^n ax} = \frac{1}{a}\left[\frac{1}{(n-3)\sin^{n-3} ax} - \frac{1}{(n-1)\sin^{n-1} ax}\right]$

$$(n \neq 1,\ n \neq 3)$$

390. $\displaystyle\int \frac{\cos^n ax}{\sin ax}\,dx = \frac{\cos^{n-1} ax}{a(n-1)} + \int \frac{\cos^{n-2} ax\,dx}{\sin ax} \qquad (n \neq 1)$

391. $\displaystyle\int \frac{\cos^n ax\,dx}{\sin^m ax} = -\frac{\cos^{n+1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-m+2}{m-1}\int \frac{\cos^n ax\,dx}{\sin^{m-2} ax}$

$$(m \neq 1)$$

$$= \frac{\cos^{n-1} ax}{a(n-m)\sin^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\cos^{n-2} ax\,dx}{\sin^m ax}$$

$$(m \neq n)$$

$$= -\frac{\cos^{n-1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-1}{m-1}\int \frac{\cos^{n-2} ax\,dx}{\sin^{m-2} ax}$$

$$(m \neq 1)$$

392. $\displaystyle\int \frac{dx}{\sin ax\,(1 \pm \cos ax)} = \pm\frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a} \log_e \tan \frac{ax}{2}$

393. $\displaystyle\int \frac{dx}{\cos ax\,(1 \pm \sin ax)} = \mp\frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a} \log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right)$

394. $\displaystyle\int \frac{\sin ax\,dx}{\cos ax\,(1 \pm \cos ax)} = \frac{1}{a} \log_e \frac{1 \pm \cos ax}{\cos ax}$

395. $\displaystyle\int \frac{\cos ax\,dx}{\sin ax\,(1 \pm \sin ax)} = -\frac{1}{a} \log_e \frac{1 \pm \sin ax}{\sin ax}$

396. $\displaystyle\int \frac{\sin ax\,dx}{\cos ax\,(1 \pm \sin ax)} = \frac{1}{2a(1 \pm \sin ax)} \pm \frac{1}{2a} \log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right)$

397. $\displaystyle\int \frac{\cos ax\,dx}{\sin ax\,(1 \pm \cos ax)} = -\frac{1}{2a(1 \pm \cos ax)} \pm \frac{1}{2a} \log_e \tan \frac{ax}{2}$

398. $\displaystyle\int \frac{\sin ax\,dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \frac{1}{2a} \log_e (\sin ax \pm \cos ax)$

399. $\displaystyle\int \frac{\cos ax\,dx}{\sin ax \pm \cos ax} = \pm\frac{x}{2} + \frac{1}{2a} \log_e (\sin ax \pm \cos ax)$

400. $\displaystyle\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}} \log_e \tan \left(\frac{ax}{2} \pm \frac{\pi}{8} \right)$

401. $\displaystyle\int \frac{dx}{1 + \cos ax \pm \sin ax} = \pm\frac{1}{a} \log_e \left(1 \pm \tan \frac{ax}{2} \right)$

402. $\displaystyle\int \frac{dx}{b \sin ax + c \cos ax} = \frac{1}{a\sqrt{b^2 + c^2}} \log_e \tan \frac{ax + \theta}{2}$

$$\sin \theta = \frac{c}{\sqrt{b^2 + c^2}} \qquad \tan \theta = \frac{c}{b}$$

403. $\displaystyle\int \frac{\sin ax\,dx}{b + c \cos ax} = -\frac{1}{ac} \log_e (b + c \cos ax)$

404. $\displaystyle\int \frac{\cos ax\,dx}{b + c \sin ax} = \frac{1}{ac} \log_e (b + c \sin ax)$

405. $\displaystyle\int \frac{\sin^2 ax\,dx}{b + c \cos^2 ax} = \frac{1}{ac} \sqrt{\frac{b + c}{b}} \arctan \left(\sqrt{\frac{b}{b + c}} \tan ax \right) - \frac{x}{c}$

406. $\displaystyle\int \frac{\sin ax \cos ax\, dx}{b \cos^2 ax + c \sin^2 ax} = \frac{1}{2a(c-b)} \log_e (b \cos^2 ax + c \sin^2 ax)$

$$(c \neq b)$$

407. $\displaystyle\int \frac{dx}{b^2 \cos^2 ax + c^2 \sin^2 ax} = \frac{1}{abc} \arctan \left(\frac{c}{b} \tan ax\right)$

408. $\displaystyle\int \frac{dx}{b^2 \cos^2 ax - c^2 \sin^2 ax} = \frac{1}{2abc} \log_e \frac{c \tan ax + b}{c \tan ax - b}$

409. $\displaystyle\int \sin ax \cos bx\, dx = - \frac{\cos (a+b)x}{2(a+b)} - \frac{\cos (a-b)x}{2(a-b)}$

$$(a^2 \neq b^2; \text{ for } a = b \text{ see } 354)$$

410. $\displaystyle\int \frac{dx}{b + c \cos ax + d \sin ax}$

$$= \begin{cases} \dfrac{-1}{a\sqrt{b^2 - c^2 - d^2}} \arcsin \dfrac{c^2 + d^2 + b(c \cos ax + d \sin ax)}{\sqrt{c^2 + d^2}(b + c \cos ax + d \sin ax)} \\ \qquad\qquad\qquad\qquad (b^2 > c^2 + d^2, |ax| < \pi) \\[2ex] \dfrac{1}{a\sqrt{c^2 + d^2 - b^2}} \\ \qquad c^2 + d^2 + b(c \cos ax + d \sin ax) \\ \log_e \dfrac{\qquad\qquad\qquad + \sqrt{c^2 + d^2 - b^2}(c \sin ax - d \cos ax)}{\sqrt{c^2 + d^2}(b + c \cos ax + d \sin ax)} \\ \qquad\qquad\qquad\qquad (b^2 < c^2 + d^2, |ax| < \pi) \\[2ex] \dfrac{1}{ab} \left[\dfrac{b - (c+d) \cos ax + (c-d) \sin ax}{b + (c-d) \cos ax + (c+d) \sin ax}\right] \quad (b^2 = c^2 + d^2) \end{cases}$$

(u) Integrals containing tangent and cotangent functions
$(a \neq 0)$

411. $\displaystyle\int \tan ax\, dx = - \frac{1}{a} \log_e \cos ax$

412. $\displaystyle\int \tan^2 ax\, dx = \frac{1}{a} \tan ax - x$

413. $\displaystyle\int \tan^3 ax\, dx = \frac{1}{2a} \tan^2 ax + \frac{1}{a} \log_e \cos ax$

414. $\displaystyle\int \tan^n ax\, dx = \frac{1}{a(n-1)} \tan^{n-1} ax - \int \tan^{n-2} ax\, dx \qquad (n > 1)$

415. $\displaystyle\int \frac{dx}{b + c \tan ax} = \int \frac{\cot ax \, dx}{b \cot ax + c}$

$$= \frac{1}{b^2 + c^2}\left[bx + \frac{c}{a} \log_e (b \cos ax + c \sin ax) \right]$$

416. $\displaystyle\int \frac{dx}{\sqrt{b + c \tan^2 ax}} = \frac{1}{a\sqrt{b - c}} \arcsin \left(\sqrt{\frac{b - c}{b}} \sin ax \right)$

$$(b > 0, \, b^2 > c^2)$$

417. $\displaystyle\int \frac{\tan^n ax}{\cos^2 ax} \, dx = \frac{\tan^{n+1} ax}{a(n + 1)} \qquad (n \neq -1)$

418. $\displaystyle\int \cot ax \, dx = \frac{1}{a} \log_e \sin ax$

419. $\displaystyle\int \cot^2 ax \, dx = \int \frac{dx}{\tan^2 ax} = -\frac{1}{a} \cot ax - x$

420. $\displaystyle\int \cot^3 ax \, dx = -\frac{1}{2a} \cot^2 ax - \frac{1}{a} \log_e \sin ax$

421. $\displaystyle\int \cot^n ax \, dx = \int \frac{dx}{\tan^n ax} = -\frac{1}{a(n - 1)} \cot^{n-1} ax$

$$- \int \cot^{n-2} ax \, dx \qquad (n > 1)$$

422. $\displaystyle\int \frac{dx}{b + c \cot ax} = \int \frac{\tan ax \, dx}{b \tan ax + c}$

$$= \frac{1}{b^2 + c^2}\left[bx - \frac{c}{a} \log_e (c \cos ax + b \sin ax) \right]$$

423. $\displaystyle\int \frac{\cot^n ax}{\sin^2 ax} \, dx = \frac{-\cot^{n+1} ax}{a(n + 1)} \qquad (n \neq -1)$

(v) Integrals containing hyperbolic functions $\qquad (a \neq 0)$

424. $\displaystyle\int \sinh x \, dx = \cosh x$

425. $\displaystyle\int \sinh^2 x \, dx = \frac{\sinh 2x}{4} - \frac{x}{2}$

426. $\displaystyle\int \frac{dx}{\sinh x} = \log_e \tanh \left(\frac{x}{2} \right)$

427. $\displaystyle\int \frac{dx}{\sinh^2 x} = -\coth x$

428. $\displaystyle\int \cosh x \, dx = \sinh x$

429. $\displaystyle\int \cosh^2 x \, dx = \frac{\sinh 2x}{4} + \frac{x}{2}$

430. $\displaystyle\int \frac{dx}{\cosh x} = 2 \arctan e^x = \arctan (\sinh x)$

431. $\displaystyle\int \frac{dx}{\cosh^2 x} = \tanh x$

432. $\displaystyle\int \frac{\sinh x}{\cosh^2 x} \, dx = -\frac{1}{\cosh x}$

433. $\displaystyle\int \frac{\cosh x}{\sinh^2 x} \, dx = -\frac{1}{\sinh x}$

434. $\displaystyle\int x \sinh x \, dx = x \cosh x - \sinh x$

435. $\displaystyle\int x \cosh x \, dx = x \sinh x - \cosh x$

436. $\displaystyle\int \tanh x \, dx = \log_e \cosh x$

437. $\displaystyle\int \tanh^2 x \, dx = x - \tanh x$

438. $\displaystyle\int \coth x \, dx = \log_e \sinh x$

439. $\displaystyle\int \coth^2 x \, dx = x - \coth x$

440. $\displaystyle\int \sinh^n ax \, dx$

$$= \begin{cases} \dfrac{1}{an} \sinh^{n-1} ax \cosh ax - \dfrac{n-1}{n} \displaystyle\int \sinh^{n-2} ax \, dx & (n > 0) \\[4mm] \dfrac{1}{a(n+1)} \sinh^{n+1} ax \cosh ax - \dfrac{n+2}{n+1} \displaystyle\int \sinh^{n+2} ax \, dx \\[2mm] \hspace{6cm} (n < -1) \end{cases}$$

441. $\displaystyle\int \cosh^n ax\, dx$

$$= \begin{cases} \dfrac{1}{an}\sinh ax \cosh^{n-1} ax + \dfrac{n-1}{n}\displaystyle\int \cosh^{n-2} ax\, dx & (n > 0) \\[3mm] -\dfrac{1}{a(n+1)}\sinh ax \cosh^{n+1} ax + \dfrac{n+2}{n+1}\displaystyle\int \cosh^{n+2} ax\, dx \\[3mm] \hspace{7cm} (n < -1) \end{cases}$$

442. $\displaystyle\int \sinh ax \sinh bx\, dx = \dfrac{\sinh (a+b)x}{2(a+b)} - \dfrac{\sinh (a-b)x}{2(a-b)}$

443. $\displaystyle\int \cosh ax \cosh bx\, dx = \dfrac{\sinh (a+b)x}{2(a+b)} + \dfrac{\sinh (a-b)x}{2(a-b)}$ $(a^2 \neq b^2)$

444. $\displaystyle\int \sinh ax \cosh bx\, dx = \dfrac{\cosh (a+b)x}{2(a+b)} + \dfrac{\cosh (a-b)x}{2(a-b)}$

445. $\displaystyle\int \sinh ax \sin ax\, dx = \dfrac{1}{2a}\left(\cosh ax \sin ax - \sinh ax \cos ax\right)$

446. $\displaystyle\int \cosh ax \cos ax\, dx = \dfrac{1}{2a}\left(\sinh ax \cos ax + \cosh ax \sin ax\right)$

447. $\displaystyle\int \sinh ax \cos ax\, dx = \dfrac{1}{2a}\left(\cosh ax \cos ax + \sinh ax \sin ax\right)$

448. $\displaystyle\int \cosh ax \sin ax\, dx = \dfrac{1}{2a}\left(\sinh ax \sin ax - \cosh ax \cos ax\right)$

(w) Integrals containing exponential functions

449. $\displaystyle\int e^{ax}\, dx = \dfrac{1}{a} e^{ax}$

450. $\displaystyle\int x e^{ax}\, dx = \dfrac{e^{ax}}{a^2}(ax - 1)$

451. $\displaystyle\int x^2 e^{ax}\, dx = e^{ax}\left(\dfrac{x^2}{a} - \dfrac{2x}{a^2} + \dfrac{2}{a^3}\right)$

452. $\displaystyle\int x^n e^{ax}\, dx = \dfrac{1}{a} x^n e^{ax} - \dfrac{n}{a}\displaystyle\int x^{n-1} e^{ax}\, dx$ $(n > 0)$

453. $\displaystyle\int \dfrac{e^{ax}}{x}\, dx = \log_e x + \dfrac{ax}{1\cdot 1!} + \dfrac{(ax)^2}{2\cdot 2!} + \dfrac{(ax)^3}{3\cdot 3!} + \cdots$

454. $\displaystyle\int \frac{e^{ax}}{x^n}\,dx = \frac{1}{n-1}\left(-\frac{e^{ax}}{x^{n-1}} + a\int \frac{e^{ax}}{x^{n-1}}\,dx\right)$ $\quad(n > 1)$

455. $\displaystyle\int \frac{dx}{1+e^{ax}} = \frac{1}{a}\log_e \frac{e^{ax}}{1+e^{ax}}$

456. $\displaystyle\int \frac{dx}{b+ce^{ax}} = \frac{x}{b} - \frac{1}{ab}\log_e (b+ce^{ax})$

457. $\displaystyle\int \frac{e^{ax}\,dx}{b+ce^{ax}} = \frac{1}{ac}\log_e (b+ce^{ax})$

458. $\displaystyle\int \frac{dx}{be^{ax}+ce^{-ax}} = \frac{1}{a\sqrt{bc}}\arctan\left(e^{ax}\sqrt{\frac{b}{c}}\right)$ $\quad(bc > 0)$

$\displaystyle\qquad\qquad = \frac{1}{2a\sqrt{-bc}}\log_e \frac{c+e^{ax}\sqrt{-bc}}{c-e^{ax}\sqrt{-bc}}$ $\quad(bc < 0)$

459. $\displaystyle\int \frac{xe^{ax}\,dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)}$

460. $\displaystyle\int e^{ax}\log_e x\,dx = \frac{1}{a}e^{ax}\log_e x - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx$

461. $\displaystyle\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

462. $\displaystyle\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

463. $\displaystyle\int xe^{ax}\sin bx\,dx = \frac{xe^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$\displaystyle\qquad\qquad - \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\sin bx - 2ab\cos bx]$

464. $\displaystyle\int xe^{ax}\cos bx\,dx = \frac{xe^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$\displaystyle\qquad\qquad - \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\cos bx + 2ab\sin bx]$

465. $\displaystyle\int e^{ax}\sin bx\sin cx\,dx = \frac{e^{ax}[(b-c)\sin (b-c)x + a\cos (b-c)x]}{2[a^2+(b-c)^2]}$

$\displaystyle\qquad\qquad - \frac{e^{ax}[(b+c)\sin (b+c)x + a\cos (b+c)x]}{2[a^2+(b+c)^2]}$

466. $\displaystyle \int e^{ax} \cos bx \cos cx \, dx = \frac{e^{ax}[(b-c)\sin(b-c)x + a\cos(b-c)x]}{2[a^2 + (b-c)^2]}$

$\displaystyle \qquad\qquad\qquad\qquad + \frac{e^{ax}[(b+c)\sin(b+c)x + a\cos(b+c)x]}{2[a^2 + (b+c)^2]}$

467. $\displaystyle \int e^{ax} \sin bx \cos cx \, dx = \frac{e^{ax}[a\sin(b-c)x - (b-c)\cos(b-c)x]}{2[a^2 + (b-c)^2]}$

$\displaystyle \qquad\qquad\qquad\qquad + \frac{e^{ax}[a\sin(b+c)x - (b+c)\cos(b+c)x]}{2[a^2 + (b+c)^2]}$

468. $\displaystyle \int e^{ax} \sin bx \sin(bx + c) \, dx$

$\displaystyle \qquad = \frac{e^{ax}\cos c}{2a} - \frac{e^{ax}[a\cos(2bx+c) + 2b\sin(2bx+c)]}{2(a^2 + 4b^2)}$

469. $\displaystyle \int e^{ax} \cos bx \cos(bx + c) \, dx$

$\displaystyle \qquad = \frac{e^{ax}\cos c}{2a} + \frac{e^{ax}[a\cos(2bx+c) + 2b\sin(2bx+c)]}{2(a^2 + 4b^2)}$

470. $\displaystyle \int e^{ax} \sin bx \cos(bx + c) \, dx$

$\displaystyle \qquad = -\frac{e^{ax}\sin c}{2a} + \frac{e^{ax}[a\sin(2bx+c) - 2b\cos(2bx+c)]}{2(a^2 + 4b^2)}$

471. $\displaystyle \int e^{ax} \cos bx \sin(bx + c) \, dx$

$\displaystyle \qquad = \frac{e^{ax}\sin c}{2a} + \frac{e^{ax}[a\sin(2bx+c) - 2b\cos(2bx+c)]}{2(a^2 + 4b^2)}$

472. $\displaystyle \int e^{ax} \sin^n bx \, dx = \frac{e^{ax}\sin^{n-1} bx \, (a\sin bx - nb\cos bx)}{a^2 + n^2 b^2}$

$\displaystyle \qquad\qquad\qquad\qquad + \frac{n(n-1)b^2}{a^2 + n^2 b^2} \int e^{ax} \sin^{n-2} bx \, dx$

473. $\displaystyle \int e^{ax} \cos^n bx \, dx = \frac{e^{ax}\cos^{n-1} bx \, (a\cos bx + nb\sin bx)}{a^2 + n^2 b^2}$

$\displaystyle \qquad\qquad\qquad\qquad + \frac{n(n-1)b^2}{a^2 + n^2 b^2} \int e^{ax} \cos^{n-2} bx \, dx$

(x) Integrals containing logarithmic functions $\quad (a \neq 0)$

474. $\displaystyle\int \log_e ax \, dx = x \log_e ax - x$

475. $\displaystyle\int (\log_e ax)^2 \, dx = x(\log_e ax)^2 - 2x \log_e ax + 2x$

476. $\displaystyle\int (\log_e ax)^n \, dx = x(\log_e ax)^n - n \int (\log_e ax)^{n-1} \, dx \qquad (n \neq -1)$

477. $\displaystyle\int \frac{dx}{\log_e ax} = \frac{1}{a}\left[\log_e (\log_e ax) + \log_e ax + \frac{(\log_e ax)^2}{2 \cdot 2!} \right.$

$$\left. + \frac{(\log_e ax)^3}{3 \cdot 3!} + \cdots \right]$$

478. $\displaystyle\int x \log_e ax \, dx = \frac{x^2}{2} \log_e ax - \frac{x^2}{4}$

479. $\displaystyle\int x^2 \log_e ax \, dx = \frac{x^3}{3} \log_e ax - \frac{x^3}{9}$

480. $\displaystyle\int x^n \log_e ax \, dx = x^{n+1}\left[\frac{\log_e ax}{n + 1} - \frac{1}{(n + 1)^2} \right] \qquad (n \neq -1)$

481. $\displaystyle\int x^n(\log_e ax)^m \, dx = \frac{x^{n+1}}{n + 1} (\log_e ax)^m - \frac{m}{n + 1} \int x^n (\log_e ax)^{m-1} \, dx$

$$(m, n \neq -1)$$

482. $\displaystyle\int \frac{(\log_e ax)^n}{x} \, dx = \frac{(\log_e ax)^{n+1}}{n + 1} \qquad (n \neq -1)$

483. $\displaystyle\int \frac{\log_e x}{x^n} \, dx = - \frac{\log_e x}{(n - 1)x^{n-1}} - \frac{1}{(n - 1)^2 x^{n-1}} \qquad (n \neq 1)$

484. $\displaystyle\int \frac{(\log_e x)^m}{x^n} \, dx = - \frac{(\log_e x)^m}{(n - 1)x^{n-1}} + \frac{m}{n - 1} \int \frac{(\log_e x)^{m-1}}{x^n} \, dx$

$$(n \neq 1)$$

485. $\displaystyle\int \frac{x^n \, dx}{\log_e ax} = \frac{1}{a^{n+1}}\left[\log_e (\log_e ax) + (n + 1) \log_e ax \right.$

$$\left. + \frac{(n + 1)^2(\log_e ax)^2}{2 \cdot 2!} + \frac{(n + 1)^3(\log_e ax)^3}{3 \cdot 3!} + \cdots \right]$$

$$= \frac{1}{a^{n+1}} \int \frac{e^y \, dy}{y} \qquad [y = (n + 1) \log_e ax]$$

486. $\displaystyle\int \frac{x^n\, dx}{(\log_e ax)^m} = \frac{-x^{n+1}}{(m-1)(\log_e ax)^{m-1}} + \frac{n+1}{m-1}\int \frac{x^n\, dx}{(\log_e ax)^{m-1}}$

$$(m \neq 1)$$

487. $\displaystyle\int \frac{dx}{x \log_e ax} = \log_e (\log_e ax)$

488. $\displaystyle\int \frac{dx}{x(\log_e ax)^n} = -\frac{1}{(n-1)(\log_e ax)^{n-1}}$

489. $\displaystyle\int \sin (\log_e ax)\, dx = \frac{x}{2}\left[\sin (\log_e ax) - \cos (\log_e ax)\right]$

490. $\displaystyle\int \cos (\log_e ax)\, dx = \frac{x}{2}\left[\sin (\log_e ax) + \cos (\log_e ax)\right]$

491. $\displaystyle\int e^{ax} \log_e bx\, dx = \frac{1}{a}\, e^{ax} \log_e bx - \frac{1}{a}\int \frac{e^{ax}}{x}\, dx$

(y) Integrals containing inverse trigonometric and hyperbolic functions $(a > 0)$

492. $\displaystyle\int \arcsin \frac{x}{a}\, dx = x \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}$

493. $\displaystyle\int x \arcsin \frac{x}{a}\, dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right) \arcsin \frac{x}{a} + \frac{x}{4}\sqrt{a^2 - x^2}$

494. $\displaystyle\int x^2 \arcsin \frac{x}{a}\, dx = \frac{x^3}{3} \arcsin \frac{x}{a} + \frac{1}{9}(x^2 + 2a^2)\sqrt{a^2 - x^2}$

495. $\displaystyle\int x^n \arcsin \frac{x}{a}\, dx = \frac{x^{n+1}}{n+1} \arcsin \frac{x}{a} - \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{a^2 - x^2}}\, dx$

$$(n \neq -1)$$

496. $\displaystyle\int \frac{\arcsin \dfrac{x}{a}\, dx}{x} = \frac{x}{a} + \frac{1}{2\cdot 3\cdot 3}\frac{x^3}{a^3}$

$$+ \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\frac{x^5}{a^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}\frac{x^7}{a^7} + \cdots$$

$$(x^2 < a^2)$$

497. $\displaystyle\int \frac{\arcsin \dfrac{x}{a}\, dx}{x^2} = -\frac{1}{x} \arcsin \frac{x}{a} - \frac{1}{a}\log_e \frac{a + \sqrt{a^2 - x^2}}{x}$

498. $\int \left(\arcsin \dfrac{x}{a} \right)^2 dx = x \left(\arcsin \dfrac{x}{a} \right)^2 + 2 \left(\sqrt{a^2 - x^2} \arcsin \dfrac{x}{a} - x \right)$

499. $\int \arccos \dfrac{x}{a} \, dx = x \arccos \dfrac{x}{a} - \sqrt{a^2 - x^2}$

500. $\int x \arccos \dfrac{x}{a} \, dx = \left(\dfrac{x^2}{2} - \dfrac{a^2}{4} \right) \arccos \dfrac{x}{a} - \dfrac{x}{4} \sqrt{a^2 - x^2}$

501. $\int x^2 \arccos \dfrac{x}{a} \, dx = \dfrac{x^3}{3} \arccos \dfrac{x}{a} - \dfrac{1}{9} (x^2 + 2a^2) \sqrt{a^2 - x^2}$

502. $\int x^n \arccos \dfrac{x}{a} \, dx = \dfrac{x^{n+1}}{n+1} \arccos \dfrac{x}{a} + \dfrac{1}{n+1} \int \dfrac{x^{n+1}}{\sqrt{a^2 - x^2}} \, dx$

$$(n \neq -1)$$

503. $\int \dfrac{\arccos \dfrac{x}{a} \, dx}{x} = \dfrac{\pi}{2} \log_e x - \dfrac{x}{a}$

$$- \dfrac{1}{2 \cdot 3 \cdot 3} \dfrac{x^3}{a^3} - \dfrac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 5} \dfrac{x^5}{a^5} - \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 7} \dfrac{x^7}{a^7} - \cdots$$

$$(x^2 < a^2)$$

504. $\int \dfrac{\arccos \dfrac{x}{a} \, dx}{x^2} = - \dfrac{1}{x} \arccos \dfrac{x}{a} + \dfrac{1}{a} \log_e \dfrac{a + \sqrt{a^2 - x^2}}{x}$

505. $\int \left(\arccos \dfrac{x}{a} \right)^2 dx = x \left(\arccos \dfrac{x}{a} \right)^2 - 2 \left(\sqrt{a^2 - x^2} \arccos \dfrac{x}{a} + x \right)$

506. $\int \arctan \dfrac{x}{a} \, dx = x \arctan \dfrac{x}{a} - \dfrac{a}{2} \log_e (a^2 + x^2)$

507. $\int x \arctan \dfrac{x}{a} \, dx = \dfrac{1}{2} (x^2 + a^2) \arctan \dfrac{x}{a} - \dfrac{ax}{2}$

508. $\int x^2 \arctan \dfrac{x}{a} \, dx = \dfrac{x^3}{3} \arctan \dfrac{x}{a} - \dfrac{ax^2}{6} + \dfrac{a^3}{6} \log_e (a^2 + x^2)$

509. $\int x^n \arctan \dfrac{x}{a} \, dx = \dfrac{x^{n+1}}{n+1} \arctan \dfrac{x}{a} - \dfrac{a}{n+1} \int \dfrac{x^{n+1} \, dx}{a^2 + x^2}$

$$(n \neq -1)$$

510. $\displaystyle\int \frac{\arctan \frac{x}{a}\, dx}{x} = \frac{x}{a} - \frac{x^3}{3^2 a^3} + \frac{x^5}{5^2 a^5} - \frac{x^7}{7^2 a^7} + \cdots \quad (|x| < |a|)$

511. $\displaystyle\int \frac{\arctan \frac{x}{a}\, dx}{x^2} = -\frac{1}{x} \arctan \frac{x}{a} - \frac{1}{2a} \log_e \frac{a^2 + x^2}{x^2}$

512. $\displaystyle\int \frac{\arctan \frac{x}{a}\, dx}{x^n} = -\frac{1}{(n-1)x^{n-1}} \arctan \frac{x}{a}$

$$+ \frac{a}{n-1} \int \frac{dx}{x^{n-1}(a^2 + x^2)} \qquad (n \neq 1)$$

513. $\displaystyle\int \operatorname{arccot} \frac{x}{a}\, dx = x \operatorname{arccot} \frac{x}{a} + \frac{a}{2} \log_e (a^2 + x^2)$

514. $\displaystyle\int x \operatorname{arccot} \frac{x}{a}\, dx = \frac{1}{2}(x^2 + a^2) \operatorname{arccot} \frac{x}{a} + \frac{ax}{2}$

515. $\displaystyle\int x^2 \operatorname{arccot} \frac{x}{a}\, dx = \frac{x^3}{3} \operatorname{arccot} \frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6} \log_e (a^2 + x^2)$

516. $\displaystyle\int x^n \operatorname{arccot} \frac{x}{a}\, dx = \frac{x^{n+1}}{n+1} \operatorname{arccot} \frac{x}{a} + \frac{a}{n+1} \int \frac{x^{n+1}\, dx}{a^2 + x^2}$

$$(n \neq -1)$$

517. $\displaystyle\int \frac{\operatorname{arccot} \frac{x}{a}\, dx}{x} = \frac{\pi}{2} \log_e x - \frac{x}{a} + \frac{x^3}{3^2 a^3} - \frac{x^5}{5^2 a^5} + \frac{x^7}{7^2 a^7} - \cdots$

518. $\displaystyle\int \frac{\operatorname{arccot} \frac{x}{a}\, dx}{x^2} = -\frac{1}{x} \operatorname{arccot} \frac{x}{a} + \frac{1}{2a} \log_e \frac{a^2 + x^2}{x^2}$

519. $\displaystyle\int \frac{\operatorname{arccot} \frac{x}{a}\, dx}{x^n} = -\frac{1}{(n-1)x^{n-1}} \operatorname{arccot} \frac{x}{a}$

$$- \frac{a}{n-1} \int \frac{dx}{x^{n-1}(a^2 + x^2)} \qquad (n \neq 1)$$

520. $\displaystyle\int \sinh^{-1}\frac{x}{a}\,dx = x\sinh^{-1}\frac{x}{a} - \sqrt{x^2+a^2}$

521. $\displaystyle\int x\sinh^{-1}\frac{x}{a}\,dx = \frac{1}{2}\left(x^2+\frac{a^2}{2}\right)\sinh^{-1}\frac{x}{a} - \frac{x}{4}\sqrt{x^2+a^2}$

522. $\displaystyle\int \cosh^{-1}\frac{x}{a}\,dx = x\cosh^{-1}\frac{x}{a} \mp \sqrt{x^2-a^2}$

$\left(\text{upper sign for } \cosh^{-1}\frac{x}{a} > 0\right)$

523. $\displaystyle\int \tanh^{-1}\frac{x}{a}\,dx = x\tanh^{-1}\frac{x}{a} + \frac{a}{2}\log_e(a^2-x^2)$

524. $\displaystyle\int x\tanh^{-1}\frac{x}{a}\,dx = \frac{x^2-a^2}{2}\tanh^{-1}\frac{x}{a} + \frac{ax}{2}$

525. $\displaystyle\int \coth^{-1}\frac{x}{a}\,dx = x\coth^{-1}\frac{x}{a} + \frac{a}{2}\log_e(x^2-a^2)$