

M251 Practice Exam for 7.8-8.6

For every problem that states "determine the convergence or divergence of the series" also **justify your answer by identifying the theorem or test and showing how the condition or conditions were satisfied.**

1. Write the first five terms of the sequence.

$$a_n = \left(-\frac{4}{5}\right)^n$$

$$-\frac{4}{5}, \frac{16}{25}, -\frac{64}{125}, \frac{256}{625}, -\frac{1024}{3125}$$

2. Write the first five terms of the sequence.

$$a_n = (-1)^{n+4} \left(\frac{17}{n}\right)$$

$$-17, \frac{17}{2}, -\frac{17}{3}, \frac{17}{4}, -\frac{17}{5}$$

3. Write the first five terms of the sequence.

$$a_n = 5 - \frac{3}{n} - \frac{7}{n^2}$$

$$-5, \frac{7}{4}, \frac{29}{9}, \frac{61}{16}, \frac{103}{25}$$

4. Determine the convergence or divergence of the sequence with the given  $n$ th term. If the sequence converges, find its limit.

$$a_n = \frac{\ln(n^{10})}{6n}$$

$\infty$  form

$\rightarrow$  L'Hopital's rule

$$= \boxed{\lim_{n \rightarrow \infty} \frac{10 \frac{1}{n}}{6} = 0 \text{ converges}}$$

5. Determine the convergence or divergence of the sequence with the given  $n$ th term. If the sequence converges, find its limit.

$$a_n = \frac{\ln(\sqrt[3]{n})}{8n} = \frac{\ln n}{9.8n} \quad \text{in form, by L'Hopital's Rule:}$$

$$\lim a_n = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{72} = 0$$

6. Determine the convergence or divergence of the sequence with the given  $n$ th term. If the sequence converges, find its limit.

$$a_n = \frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n \rightarrow 0$$

7. Write the first five terms of the sequence of partial sums.

$$5 + \frac{5}{4} + \frac{5}{9} + \frac{5}{16} + \frac{5}{25} + \dots \quad S_1 = 5 \\ S_2 = 5 + \frac{5}{4} = 6.25 \\ S_3 = S_2 + \frac{5}{9} = \frac{245}{36} \\ S_4 = \frac{1025}{144} \quad S_5 = \frac{5269}{720}$$

8. Write the first five terms of the sequence of partial sums.

$$-5 + \frac{25}{6} - \frac{125}{36} + \frac{625}{216} - \frac{3125}{1296} + \dots \quad S_1 = -5 \\ S_2 = -5/6 \\ S_3 = -155/36 \\ S_4 = -305/216 \\ S_5 = -4955/1296$$

9. Write the first five terms of the sequence of partial sums.

$$\sum_{n=1}^{\infty} \frac{5}{(4)^{n-1}} \quad S_1 = 5 \\ S_2 = S_1 + \frac{5}{4} = 6.25 \\ S_3 = S_2 + \frac{5}{16} = \frac{105}{16} \quad S_4 = S_3 + \frac{5}{64} = 425/64 \\ S_5 = S_4 + \frac{5}{256} = 1705/256$$

10. Find the sum of the convergent series.

$$\sum_{n=1}^{\infty} \frac{6}{(n+4)(n+6)} = \frac{A}{n+4} + \frac{B}{n+6}$$

$$6 = (n+6)A + (n+4)B$$

11. Find the sum of the convergent series.

$$\sum_{n=1}^{\infty} (-1)^n \frac{4}{(n+9)(n+11)} = \frac{A}{n+9} + \frac{B}{n+11}$$

$$4 = (n+11)A + (n+9)B$$

12. Find the sum of the convergent series.

$\sum_{n=0}^{\infty} 9\left(\frac{10}{11}\right)^n$  geometric series with  $r = \frac{10}{11} < 1$   
 converges to  $\frac{\text{first}}{1-\text{ratio}} = \frac{9}{1-\frac{10}{11}} = 99$

13. Find the sum of the convergent series.

$\sum_{n=0}^{\infty} 2\left(-\frac{9}{10}\right)^n$  geometric series with  $-1 < r = -\frac{9}{10} < 1$   
 converges to  $\frac{\text{first}}{1-\text{ratio}} = \frac{2}{1-\left(-\frac{9}{10}\right)} = \frac{20}{19}$

14. Determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \frac{4^n}{n^4} = \sum_{n=1}^{\infty} \frac{n^4}{4^n}$$

using the ratio test we have.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^4}{4^{n+1}}}{\frac{n^4}{4^n}} = \lim_{n \rightarrow \infty} \frac{(n+1)^4}{4} \cdot \frac{4^n}{4^{n+1}} = \frac{1}{4}$$

15. Determine the convergence or divergence of the series.

$$\sum_{n=0}^{\infty} \frac{2}{2^n}$$

This is a geometric series with  $r = \frac{1}{2} \Rightarrow |r| < 1 \Rightarrow$  series converges

16. Find all values of  $x$  for which the series converges. For these values of  $x$ , write the sum of the series as a function of  $x$ .

$\sum_{n=0}^{\infty} \frac{x^n}{9^n}$  This is a geometric series with  $r = \frac{x}{9}$  which converges for  $-1 < \frac{x}{9} < 1 \text{ i.e. } -9 < x < 9$ . For those values of  $x$  the series converges to  $\frac{\text{first}}{1-\text{ratio}} = \frac{1}{1-\frac{x}{9}} = \frac{9}{9-x}$

17. Find all values of  $x$  for which the series converges. For these values of  $x$ , write the sum of the series as a function of  $x$ .

$\sum_{n=0}^{\infty} 10\left(\frac{x-4}{10}\right)^n$  This is a geometric series with  $r = \frac{x-4}{10}$  which converges for  $-1 < \frac{x-4}{10} < 1 \Rightarrow -10 < x-4 < 10 \Rightarrow -6 < x < 14$ . For those values the series converges to  $\frac{\text{first}}{1-\text{ratio}} = \frac{10}{1-\frac{x-4}{10}} = \frac{100}{14-x}$

18. Use the Integral Test to determine the convergence or divergence of the series. Show your work.

Let  $f(x) = \frac{7}{10x+2}$  then  $f'(x) = \frac{-70}{(10x+2)^2} < 0$   
 $\sum_{n=1}^{\infty} \frac{7}{10n+2}$  so  $f(x)$  is differentiable, continuous, and decreasing  $\Rightarrow$  series and  $\int f(x)dx$  converge or diverge together.  
 $\int_1^{\infty} \frac{7}{10x+2} dx = \lim_{b \rightarrow \infty} \left[ \frac{7}{10} \ln(10x+2) \right]_1^b = \frac{7}{10} \left( \lim_{b \rightarrow \infty} (10b+2) - \ln 12 \right) = \infty$   
So the series diverges

19. Use the Integral Test to determine the convergence or divergence of the series. Show your work.

Let  $f(x) = x e^{-\frac{x}{2}}$  then  $f'(x) = e^{-\frac{x}{2}} - \frac{x}{2} e^{-\frac{x}{2}} = \left(1 - \frac{x}{2}\right) e^{-\frac{x}{2}} < 0$   
 $\sum_{n=1}^{\infty} n e^{-\frac{n}{2}}$  so  $f(x)$  is continuous and decreasing.  
 Thus by the integral test the series converges or diverges with  $\int_1^{\infty} x e^{-\frac{x}{2}} dx$  integrating by parts  $u = x$   $dv = e^{-\frac{x}{2}} dx$   
 $b$   $du = dx$   $v = -2 e^{-\frac{x}{2}}$

20. Use the Integral Test to determine the convergence or divergence of the series. Show your work.

Let  $f(x) = \frac{\ln x}{x^2}$ ,  $f'(x) = \frac{\frac{1}{x} x^2 - 2x \ln x}{x^4} = \frac{1 - 2 \ln x}{x^3} < 0$

 $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$  so  $f(x)$  is continuous and decreasing.  
 $\int \frac{\ln x}{x^2} dx$  integrating by parts  $u = \ln x$   $dv = \frac{dx}{x^2}$   
 $du = \frac{dx}{x}$   $v = -\frac{1}{x}$ 
 $= -\frac{\ln x}{6x^6} - \int \frac{dx}{6x^7} = -\frac{\ln x}{6x^6} - \frac{1}{36x^6}$  For the limit, use L'Hopital's rule once to get  $\frac{-2}{2e^2} \rightarrow 0$   
 $\lim_{b \rightarrow \infty} \left[ -\frac{2(\ln x + 2)}{e^2} \right]_1^b$   
 So the series converges

21. Use the Integral Test to determine the convergence or divergence of the series. Show your work.

$f(x) = \frac{4}{x\sqrt{\ln x}}$  is continuous and decreasing.  
 $\sum_{n=2}^{\infty} \frac{4}{n\sqrt{\ln n}}$   $\int \frac{4}{x\sqrt{\ln x}} dx$  substitute  $u = \ln x$ ,  $du = \frac{dx}{x}$

 $\int \frac{4}{\sqrt{u}} du = 2\sqrt{u} = 2\sqrt{\ln x}; \int_2^{\infty} \frac{4}{x\sqrt{\ln x}} dx = \lim_{b \rightarrow \infty} \left[ 2\sqrt{\ln x} \right]_2^b \rightarrow \infty$   
 So the series diverges

22. Use the p-series theorem to determine the convergence or divergence of the series.

$\sum_{n=1}^{\infty} \frac{8}{n^{\frac{10}{7}}} = 8 \sum_{n=1}^{\infty} \frac{1}{n^{\frac{10}{7}}}$  which is a p-series with  $p = \frac{10}{7} > 1$

so the series converges.

23. Use the p-series theorem to determine the convergence or divergence of the series.

$1 + \frac{1}{\sqrt[3]{2^2}} + \frac{1}{\sqrt[3]{3^2}} + \frac{1}{\sqrt[3]{4^2}} + \frac{1}{\sqrt[3]{5^2}} + \dots = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$  which is a p-series  
 with  $p = \frac{2}{3} < 1$  so the series diverges

24. Use the p-series theorem to determine the convergence or divergence of the series.

$\sum_{n=1}^{\infty} \frac{1}{n^{0.78}}$  This is a p-series with  $p = 0.78 < 1$  so the series diverges

25. Determine the convergence or divergence of the series.

clearly  $1 < 3n^2 \Rightarrow 0 < 3n^2 - 1 \Rightarrow n^2 < 4n^2 - 1$

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \Rightarrow \frac{1}{4n^2 - 1} < \frac{1}{n^2} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$
 is a convergent p-series ( $p=2 > 1$ ) so by D.C.T.  $\sum \frac{1}{4n^2 - 1}$  converges

Using the LCT with  $\sum \frac{1}{n^2}$   $\lim_{n \rightarrow \infty} \frac{\frac{1}{n^2}}{\frac{1}{4n^2 - 1}} = \lim \frac{4n^2 - 1}{n^2} = \lim 4 - \frac{1}{n^2} = 4$ .

26. Determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \frac{7}{n \cdot \sqrt[3]{n}} = 7 \sum_{n=1}^{\infty} \frac{1}{n^{1.125}}$$
 which is

a p-series with  $p = \frac{9}{8} > 1$  so

the series converges

27. Determine the convergence or divergence of the series.

$3 \cdot \sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$  this is a p-series with  $p = 0.95 < 1$  so the series diverges.

28. Determine the convergence or divergence of the series.

$\sum_{n=0}^{\infty} \left(\frac{5}{3}\right)^n$  This is a geometric series with  $r = \frac{5}{3} > 1$  so the series diverges.

29. Use the Ratio Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^n \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1) \left(\frac{3}{4}\right)^{n+1}}{n \left(\frac{3}{4}\right)^n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \frac{3}{4} = \frac{3}{4}$$

$\frac{3}{4} < 1 \Rightarrow$  series converges

30. Use the Ratio Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \frac{n^9}{4^n} \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^9 4^{n+1}}{n^9 4^n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^9 4 = 4$$

$4 > 1$  so the series diverges

31. Use the Ratio Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \left(\frac{10}{8}\right)^n}{n^2} \quad \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^n \left(\frac{10}{8}\right)^{n+1}}{(n+1)^2}}{\frac{(-1)^{n-1} \left(\frac{10}{8}\right)^n}{n^2}} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^2 \frac{10}{8} = \frac{10}{8}$$

$\frac{10}{8} > 1 \Rightarrow$  series diverges

32. Use the Root Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \left(\frac{8n}{8n+1}\right)^n \quad \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \left(\frac{8n}{8n+1}\right) = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{8n}}\right) = 1$$

so the series diverges.

33. Use the Root Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \left( \frac{4n+1}{8n-1} \right)^n \quad \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{4n+1}{8n-1} = \lim_{n \rightarrow \infty} \frac{4 + \frac{1}{n}}{8 - \frac{1}{n}} = \frac{4 + 0}{8 - 0} = \frac{1}{2} < 1$$

$\Rightarrow$  series converges

34. Use the Root Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \left( \frac{7n^2+1}{4n^2-1} \right)^n \quad \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{7n^2+1}{4n^2-1} = \lim_{n \rightarrow \infty} \frac{7 + \frac{1}{n^2}}{4 - \frac{1}{n^2}} = \frac{7 + 0}{4 - 0} = \frac{7}{4} > 1$$

$\Rightarrow$  series diverges

35. Determine the convergence or divergence of the series using any appropriate test from this chapter. Identify the test used.

From the alternating series test  
 $\sum_{n=1}^{\infty} \frac{(-1)^n 8}{3n}$  If  $f(x) = \frac{8}{3x}$  then  $f'(x) = \frac{-8}{3x^2} < 0$   
 $f' < 0 \Rightarrow$  terms are decreasing;  $\lim_{n \rightarrow \infty} \frac{8}{3n} = 0$

$\therefore$  this series converges by the A.S.T.

36. Determine the convergence or divergence of the series using any appropriate test from this chapter. Identify the test used.

$\sum_{n=1}^{\infty} \frac{6}{n^2} = 6 \sum_{n=1}^{\infty} \frac{1}{n^2}$  which is a p-series with  $p = 2 > 1$   
 hence it converges by p-series test

or using the integral test  $f(x) = \frac{6}{x^2}$  is  
 diff. and decreasing on  $x \geq 1$ . Furthermore

$$\int_1^{\infty} \frac{6}{x^2} dx = \lim_{b \rightarrow \infty} \left[ \frac{-6}{x} \right]_1^b = 6 \text{ so the series converges.}$$