

M250 Practice Test Answer Key

chap 3.1-3.6

1. Find all open intervals on which the function $f(x) = \frac{x}{x^2+x-2}$ is decreasing.

Find open intervals on which the derivative is negative.

$$\begin{aligned} f'(x) &= \frac{(x)'(x^2+x-2) - x(x^2+x-2)'}{(x^2+x-2)^2} \\ &= \frac{x^2+x-2 - x(2x+1)}{(x-1)^2(x+2)^2} \\ &= \frac{-x^2-2}{(x-1)^2(x+2)^2} = -\frac{x^2+2}{(x-1)^2(x+2)^2} \end{aligned}$$

Answer: $(-\infty, -2)$, $(-2, 1)$ and $(1, \infty)$

2. Find all critical numbers for the function

$$f(x) = (9-x^2)^{3/5}$$

$$f'(x) = \frac{3}{5}(9-x^2)^{-2/5} \cdot (-2x) = \frac{-6x}{\sqrt[5]{(9-x^2)^2}}$$

$f'(0) = 0$, $f'(3)$ does not exist, $f'(-3)$ does not exist

Answer: $\{-3, 0, 3\}$

3. Find the values of x that give relative extrema for the function
 $f(x) = (x+1)^2(x-2)$

$$\begin{aligned} f'(x) &= 2(x+1)(x-2) + (x+1)^2 \\ &= (x+1)(2x-4+x+1) \\ &= (x+1)(3x-3) \\ &= 3(x+1)(x-1) \end{aligned}$$

critical numbers are -1 and 1

$$\begin{aligned} f''(x) &= 3(x-1) + 3(x+1) \\ &= 6x \end{aligned}$$

$$f''(-1) = -6 < 0 \text{ cupped downward}$$

Answer: \rightarrow relative max. at $x = -1$

$$\begin{aligned} f''(1) &= 6 > 0 \text{ cupped upward} \\ &\rightarrow \text{relative min. at } x = 1 \end{aligned}$$

4. Find all intervals on which the graph of the function is concave upward

$$f(x) = \frac{x-1}{x+3}$$

ie. where is $f'' > 0$?

$$\begin{aligned} f'(x) &= \frac{(x-1)'(x+3) - (x-1)(x+3)'}{(x+3)^2} \\ &= \frac{x+3 - x+1}{(x+3)^2} = \frac{4}{(x+3)^2} \end{aligned}$$

4. (continued)

$$f'(x) = 4(x+3)^{-2}$$

$$f''(x) = -8(x+3)^{-3} = \frac{-8}{(x+3)^3}$$

so $f''(x) > 0$ whenever $(x+3)^3 < 0$

i.e. $x+3 < 0 \Rightarrow$ on $(-\infty, -3)$

Answer:

$f(x)$ is concave upward on $(-\infty, -3)$

5. Let $f''(x) = 3x^2 - 4$ and let $f(x)$ have critical numbers -2 , 0 , and 2 . Use the Second Derivative Test to determine which critical numbers, if any, give a relative maximum.

$$f''(-2) = 3(-2)^2 - 4 = 8 > 0 \Rightarrow f(-2) \text{ is a rel. min.}$$

$$f''(0) = 3(0)^2 - 4 = -4 < 0 \Rightarrow f(0) \text{ is a rel. max.}$$

$$f''(2) = 3(2)^2 - 4 = 8 > 0 \Rightarrow f(2) \text{ is a rel. min.}$$

Answer: $x=0$ gives a relative maximum.

6. Find $\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2-1}}{x^2}$

$$\frac{\sqrt{4x^2-1}}{x^2} = \frac{\sqrt{4x^2-1}}{\sqrt{x^4}} = \sqrt{\frac{4x^2-1}{x^4}} = \sqrt{\frac{4}{x^2} - \frac{1}{x^4}}$$

$$\lim_{x \rightarrow \infty} \sqrt{\frac{4}{x^2} - \frac{1}{x^4}} = \sqrt{0-0} = 0$$

7. Which of the following functions has a horizontal asymptote at $y = -\frac{1}{2}$?

a) $\lim_{x \rightarrow \pm\infty} \frac{x^3}{1-2x^3} = \lim_{x \rightarrow \pm\infty} \frac{\frac{x^3}{x^3}}{\frac{1-2x^3}{x^3}} = \lim_{x \rightarrow \pm\infty} \frac{1}{\frac{1}{x^3}-2} = -\frac{1}{2}$

So $\frac{x^3}{1-2x^3}$ has a horizontal asymptote at $y = -\frac{1}{2}$

b) $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{2x+1}} = \lim_{x \rightarrow \infty} \frac{\frac{x}{\sqrt{x}}}{\frac{\sqrt{2x+1}}{\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\sqrt{2+\frac{1}{x}}} = \infty$

So $\frac{x}{\sqrt{2x+1}}$ has no horizontal asymptotes

c) $\lim_{x \rightarrow \pm\infty} \frac{2x^2-6x+1}{1+x^2} = \lim_{x \rightarrow \pm\infty} \frac{\frac{2x^2-6x+1}{x^2}}{\frac{1+x^2}{x^2}} =$

$$= \lim_{x \rightarrow \pm\infty} \frac{2 - \frac{6}{x} + \frac{1}{x^2}}{\frac{1}{x^2} + 1} = \frac{2-0+1}{0+1} = 2$$

So $\frac{2x^2-6x+1}{1+x^2}$ has a horizontal asymptote at $y = 2$

d) $\lim_{x \rightarrow \pm\infty} \frac{x-1}{2x^2+1} = \lim_{x \rightarrow \pm\infty} \frac{\frac{x-1}{x^2}}{\frac{2x^2+1}{x^2}} = \lim_{x \rightarrow \pm\infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{2 + \frac{1}{x^2}} = \frac{0-0}{2+0} = 0$

So $\frac{x-1}{2x^2+1}$ has a horizontal asymptote of $y = 0$.

8. Sketch the graph of $f(x) = \frac{x-1}{x+2}$

vertical asymptote: $x = -2$

horizontal asymptote: $y = 1$

y-intercept: $(0, -\frac{1}{2})$

x-intercept: $(1, 0)$

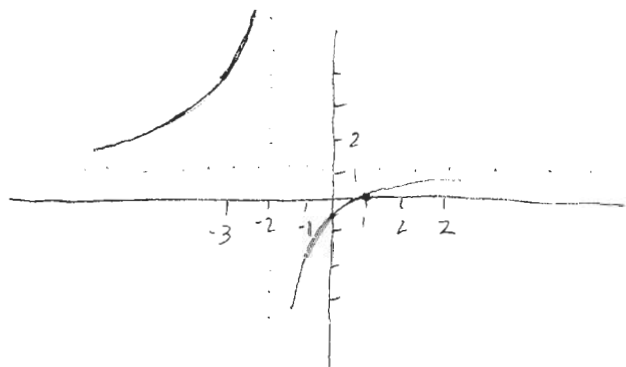
$$f'(x) = \frac{(x-1)'(x+2) - (x-1)(x+2)'}{(x+2)^2} = \frac{x+2 - (x-1)}{(x+2)^2} = \frac{3}{(x+2)^2}$$

so $f'(x) > 0$ for $x \neq -2 \Rightarrow f(x)$ is increasing
on $(-\infty, -2)$ and $(-2, \infty)$

$$f''(x) = 3(-2)(x+2)^{-3}(1) = \frac{-6}{(x+2)^2}$$

$f''(x) > 0$ on $(-\infty, -2)$ so $f(x)$ concave up on $(-\infty, -2)$

$f''(x) < 0$ on $(-2, \infty)$ so $f(x)$ concave down on $(-2, \infty)$



9. Find all points of inflection for $f(x) = x^3 - 12x$

$$f'(x) = 3x^2 - 12$$

$f''(x) = 6x$ so $f''(x)$ changes sign at
 $x = 0$

10. State the Mean Value Theorem

For $f(x)$ continuous on $[a, b]$ and
differentiable on (a, b) :

there exists c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$