The Function $\dfrac{\sin x}{x}$

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Calculus students are well aware of an important use of the function $\sin x/x$. The common and perhaps the easiest approach to finding the derivatives of the trigonometric functions hinges on the result

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{1}$$

However, few students realize that this function plays a key role in many areas of mathematics and its applications. In this paper, we briefly describe several of these roles after first presenting some elementary examples in which $\sin x/x$ arises geometrically. These examples are accessible to calculus students and may help motivate interest in and draw attention to this function. In addition, they provide some unusual proofs of the limit (1).

Properties of sin x/x

In certain fields, such as signal analysis, the function $\sin x/x$ is often called the "sinc function." Thus, let us set

$$\operatorname{sinc} x = \frac{\sin x}{x}.$$

By defining $\operatorname{sinc} 0 = 1$, the function is extended to an analytic function on the real line. We shall refer to this extension also by the name sinc. The graph of sinc is shown in Figure 1, where we note that it is an even function with roots at $n\pi$, for $n = \pm 1, \pm 2, \ldots$, that $|\operatorname{sinc} x| < 1$ for all $x \neq 0$, and that $\lim_{x \to \pm\infty} \operatorname{sinc} x = 0$.
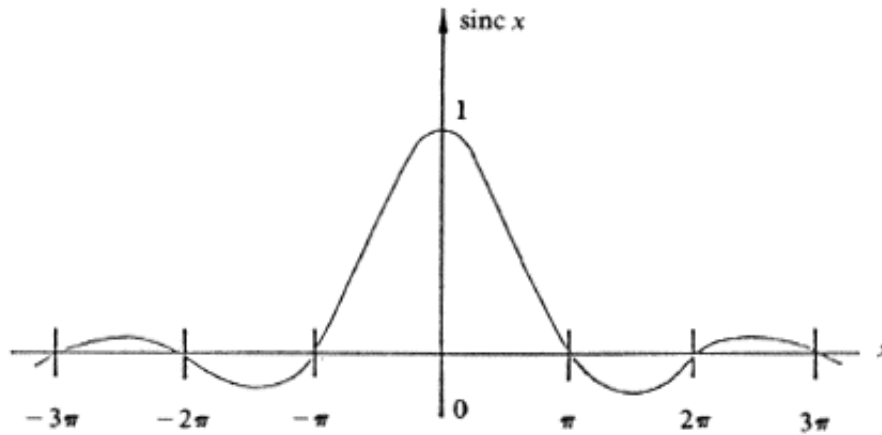
Figure 1

The sinc function has a rapidly convergent power series representation,

$$\text{sinc } x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n + 1)!},$$

and even an infinite product representation [**7**],

$$\text{sinc } x = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2} \right).$$

Also,

$$\int_{-\infty}^{\infty} \text{sinc } x \, dx = \pi,$$

although this integral is not absolutely convergent. However, the square of sinc is integrable on the real line with

$$\int_{-\infty}^{\infty} \text{sinc}^2 x \, dx = \pi.$$

All things considered, sinc is a rather nice, well-behaved function. Properties of sinc are presented in [**4**, sections 39–41] and [**2**, pages 62–63].

Examples from Geometry

In this section we present a few simple examples from geometry involving $\sin x/x$. In the process, some nonroutine proofs of the limit (1) will fall out.

Area of a Circle. In a circle of radius 1, inscribe a polygon composed of n isosceles triangles with vertex angle α and one isosceles triangle with vertex angle β where $0 \leq \beta < \alpha < \pi$ (Figure 2). Note that $\beta = 2\pi - n\alpha$ and that $\lim_{a \to 0^+} \beta = 0$. The area of an isosceles triangle with unit sides and vertex angle θ is $(1/2)\sin \theta$. It follows
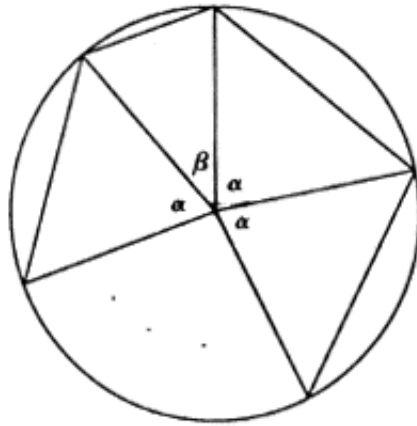
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Figure 2

that the area of the polygon is

$$A(\alpha) = n\frac{1}{2}\sin\alpha + \frac{1}{2}\sin\beta$$

$$= \frac{2\pi - \beta}{\alpha}\frac{1}{2}\sin\alpha + \frac{1}{2}\sin\beta$$

$$= \frac{2\pi - \beta}{2}\operatorname{sinc}\alpha + \frac{1}{2}\sin\beta.$$

Hence, a proof of (1) follows from

$$\pi = \lim_{\alpha \to 0^+} A(\alpha) = \frac{2\pi - 0}{2}\lim_{\alpha \to 0}\operatorname{sinc}\alpha + 0.$$

Further, the perimeter of the polygon is

$$P(\alpha) = n\left(2\sin\frac{\alpha}{2}\right) + 2\sin\frac{\beta}{2} = \left(\frac{2\pi - \beta}{\alpha}\right)\left(2\sin\frac{\alpha}{2}\right) + 2\sin\frac{\beta}{2}$$

$$= (2\pi - \beta)\operatorname{sinc}\frac{\alpha}{2} + 2\sin\frac{\beta}{2}.$$

As α approaches zero, the perimeter approaches 2π, giving us another verification of the limit (1).

It is interesting to note that some early attempts to measure π were based on measuring the perimeter of a regular polygon inscribed in a circle. For a regular polygon with n sides, $\alpha = 2\pi/n$, and $\beta = 0$. From the last formula, the perimeter is $2\pi\operatorname{sinc}(\pi/n)$. Thus, the early mathematicians were actually using $\pi\operatorname{sinc}(\pi/n)$ as an approximation of π.

Centroid of a sector.   The centroid, or center of gravity, of a plane region is the point at which a fulcrum should be placed so that the region balances. Consider a circular sector having radius 1 and central angle 2α, drawn in Figure 3 so that the vertical axis is

3

the angle bisector. The centroid of the circular sector lies on this vertical axis. The moment about the horizontal axis through the vertex is given by

$$\int_{-\sin \alpha}^{\sin \alpha} \left(\int_{x \cot \alpha}^{\sqrt{1-x^2}} y \, dy \right) dx = \frac{2}{3} \sin \alpha.$$
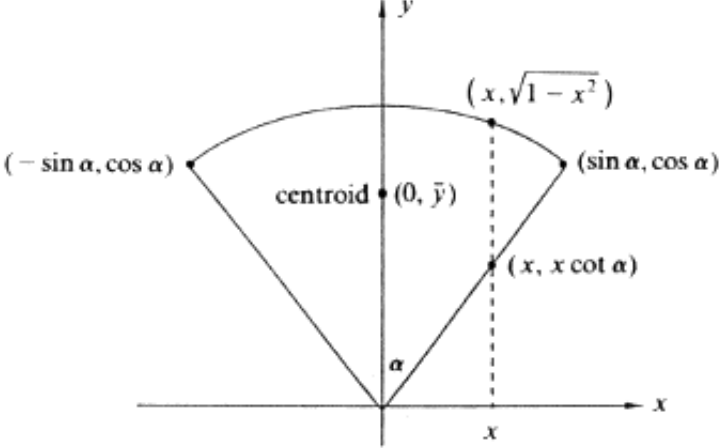

Figure 3

Since the area of the sector is α, the distance to the centroid from the vertex is

$$\bar{y} = \frac{(2/3) \sin \alpha}{\alpha} = \frac{2}{3} \operatorname{sinc} \alpha.$$

We can now give a physical argument to justify the limit (1). Suppose two triangles are drawn as illustrated in Figure 4. The centroid of an isosceles triangle is on its altitude $2/3$ of the way from the vertex. Therefore, the centroid of the larger triangle is $2/3$ units above the vertex, and the centroid of the smaller triangle is $(2/3)\cos \alpha$ units above the vertex. Since the centroid of the sector must lie between those of the triangles, we have

$$\frac{2}{3} \cos \alpha \le \frac{2}{3} \operatorname{sinc} \alpha \le \frac{2}{3}.$$
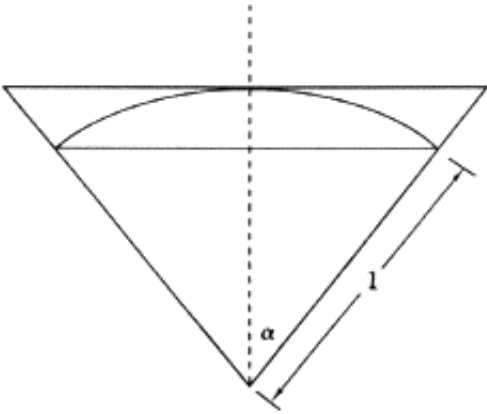

Figure 4

4

Thus, allowing α to approach zero gives the result (1).

Another example involving sinc and centroids appears in the recent paper of Lo Bello [**10**]. Consider an n-sided pyramid whose base is a regular n-sided polygon inscribed in a circle of radius r and whose peak is at a height h above the center of the circle. As shown by Lo Bello [**10**, p. 169], the volume of this pyramid and the moment about its base are

$$V = \frac{1}{3}\pi r^2 h \text{ sinc } \frac{2\pi}{n} \qquad \text{and} \qquad M_{xy} = \frac{1}{12}\pi r^2 h^2 \text{ sinc } \frac{2\pi}{n}.$$

Note that as n approaches infinity, the formulas become those for the cone.

Buffon's needle problem. In Buffon's needle problem, a needle L units long is thrown at random onto a plane ruled with horizontal parallel lines d units apart (Figure 5). Assume $L \leq d$ and let α be a fixed angle such that $0 < \alpha \leq \pi/2$. We seek the probability that the needle crosses a line, given that it lies within an angle α of the vertical. (In the usual case of Buffon's needle problem, $\alpha = \pi/2$.)
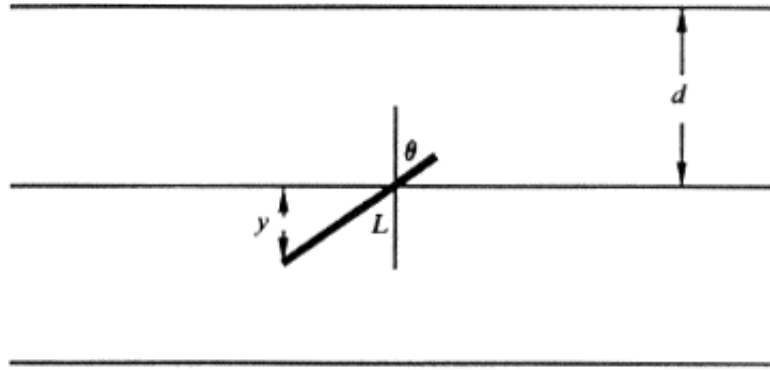

Figure 5

Let C be the event the needle crosses a line, and let $A\ (\alpha)$ be the event the needle lies within an angle α of the vertical. If y is the distance of the southernmost point of the needle from the line immediately above, and θ is the angle the needle makes with the vertical, then in order for the event $C \cap A(\alpha)$ to occur we must have both

$$y \leq L \cos \theta \qquad \text{and} \qquad -\alpha \leq \theta \leq \alpha.$$

Thus,

$$P\{C \cap A(\alpha)\} = \frac{\int_{-\alpha}^{\alpha} \int_{0}^{L \cos\theta} dy\, d\theta}{\int_{-\pi/2}^{\pi/2} \int_{0}^{d} dy\, d\theta} = \frac{1}{\pi d} \int_{-\alpha}^{\alpha} L \cos \theta d\theta = \frac{2L}{\pi d} \sin \alpha.$$

But, the probability of the event $A(\alpha)$ is $2\alpha/\pi$. Hence, the conditional probability of the event C, given that the event $A(\alpha)$ occurs, is

$$P\{C|A(\alpha)\} = \frac{\dfrac{2L}{\pi d} \sin \alpha}{\dfrac{2\alpha}{\pi}} = \frac{L}{d} \text{ sinc } \alpha.$$

If $d = L$, then the probability that the needle crosses a line, given that it lies within an angle α of the vertical, is simply sinc α. As α approaches 0, this probability must approach 1. Thus, we have a probabilistic interpretation of the limit (1). For this argument to be a proof, however, one must evaluate the integral above without implicitly using the limit (1). The article by Cipra [5] indicates a way of finding $\int \cos \theta \, d\theta$ without knowing the limit.

Expressions for π

The sinc function has played a role in the search for analytical expressions for the universal constant π. The recent article by Castellanos [3] gives a fascinating review of the quest. One of the first expressions for π, due to Francois Vieta in 1593, can be obtained by observing that

$$\text{sinc } x = \frac{\sin x}{x} = \cos\frac{x}{2}\frac{\sin\frac{x}{2}}{\frac{x}{2}} = \cos\frac{x}{2}\text{ sinc}\frac{x}{2} = \cos\frac{x}{2}\cos\frac{x}{4}\text{ sinc}\frac{x}{4}.$$

and so, for arbitrary n,

$$\text{sinc } x = \cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^n}\text{ sinc}\frac{x}{2^n}.$$

Using the limit (1), we get

$$\text{sinc } x = \prod_{n=1}^{\infty}\cos\frac{x}{2^n},$$

which is known as Euler's formula. Now, we take $x = \pi/2$, and use the half-angle formula for cosine to obtain the expression

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}\cdots.$$

This is the result due to Vieta [3, p. 69]. Evaluating this expression would be a good calculator exercise, but unfortunately the convergence is very slow.

The following formula for π also was obtained by Euler:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

To get this result, Euler equated the coefficient of x^2 in the power series for sinc,

$$\text{sinc } x = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

to the coefficient of x^2 in the product expansion

$$\text{sinc } x = \left(1 - \frac{x^2}{1^2\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots.$$

Euler obtained yet another formula for π in this way by equating coefficients of the fourth powers of x in these expansions [3, p. 75].

Some Roles of sin x/x

We mention now a few of the classical ways in which sinc occurs in mathematics and its applications. This survey, together with the references, will provide the reader with direction for further study.

Fourier analysis. The sinc function appears frequently in Fourier analysis, a major technique in the solution of differential equations and in other applications of mathematics. For a periodic function f that has period 2π and satisfies other reasonable conditions [**4**, page 90], Fourier analysis allows us to expand the function in a trigonometric series of the form

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdot \cdot \cdot. \qquad (2)$$

where the a_i's and the b_i's are constants that can be determined.

A function on the real line that is nonperiodic cannot be expressed as a trigonometric series. However, the Fourier integral formula gives an expression for such a function involving integrals of sines and cosines (rather than sums) that converges to the function at points of continuity. To derive the Fourier integral formula, we first need the following result, which can be found in [**6**, p. 78]; If f is a function on the real line that is continuous at the point x and satisfies certain other reasonable conditions, then for any positive number a,

$$f(x) = \lim_{s\to\infty} \int_{x-a}^{x+a} f(t)\frac{s}{\pi} \operatorname{sinc} s\,(t-x)dt. \qquad (3)$$

This result leads immediately to the Fourier integral theorem as follows. First note that

$$\frac{1}{\pi}\int_0^s \cos \omega\, x\, d\,\omega = \frac{\sin\omega\, x}{\pi x}\bigg|_0^s = \frac{s}{\pi} \operatorname{sinc} s\, x.$$

We can then write

$$f(x) = \lim_{s\to\infty} \int_{x-a}^{x+a} f(t)\frac{1}{\pi}\int_0^s \cos\left[\omega(t-x)\right] d\omega\, dt.$$

Changing the order of integration and extending the integration with respect to t from $-\infty$ to ∞ [**6**, p. 78], we get

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\left[\omega(t-x)\right] dt\, d\omega,$$

which is the Fourier integral formula. This derivation follows [**6**]; see also [**4**] and [**1**], particularly for proofs of (3). The Fourier integral formula can also be written in exponential form [**1**]:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)e^{-i\omega(t-x)}dt\, d\omega,$$

where i is the imaginary number $\sqrt{-1}$ and $e^{it} = \cos t + i \sin t$. We can then write

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega)e^{i\omega x}\, d\omega,$$

where

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt.$$

This last integral is called the Fourier transform of f.

From formula (3) we see that the sinc function lies at the core of the Fourier integral formula. However, sinc appears in other ways in Fourier analysis. A noteworthy example is its role in smoothing Fourier series to improve convergence and treat Gibbs' phenomenon. Gibbs' phenomenon is the tendency of a truncated Fourier series to display oscillations near points of discontinuity. Smoothing or averaging is done to reduce this effect. To smooth a Fourier series truncated to n terms, we simply multiply the k th term by the factor sinc $(k\pi/n)$ [**9**, pages 530-538].

As noted by Courant and Hilbert [**6**, p. 78], the property of sinc expressed in formula (3) was discovered by Dirichlet, and this formula is often used as the foundation of the theory of Fourier series. In certain fields where Fourier methods are fundamental, such as image processing, sinc is appropriately known as the Dirichlet function.

Spectral theory.   The appearance of sinc in Fourier analysis explains its occurrence in areas of engineering that rely on the analysis of signals, such as communication theory and acoustics. Mathematically, a signal is simply a function of time. In practice, a signal may actually be a voltage or current, which in turn represents another physical quantity such as sound energy. For a periodic signal f with Fourier series (2), we can view the signal as "decomposed" into periodic functions with discrete frequencies 0, 1, 2, . . ., and "amplitudes" given by the absolute values of the coefficients a_0, a_1, b_1,. . . . For a nonperiodic signal f on the real line we must allow every frequency ω. In the Fourier integral representation

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\,d\omega,$$

we can view f as decomposed into a continuous "spectrum" of periodic functions of x, $F(\omega)e^{i\omega x}$, for the various frequencies ω. The analogue of the amplitudes in (2) is $|F(\omega)|$, the absolute value of the Fourier transform. One can interpret $|F(\omega)|$ as an "index of the relative amplitudes of the components of the frequency spectrum of f" [**11**, p. 181]. In fact, $|F(\omega)|^2$, called the power spectrum of f at ω [**2**], [**9**], measures the intensity of the component with frequency ω.

A fundamental signal is the rectangular pulse which has the form

$$f_h(t) = \begin{cases} S/2h & \text{if } -h < t < h \\ 0 & \text{otherwise} \end{cases}$$

for positive constants S and h. The integral over time of a force acting on a system is a measure of the strength of the force. Hence, this rectangular pulse represents a force of strength S applied for a time $2h$. The Fourier transform of $f_h(t)$ is

$$F_h(\omega) = \int_{-h}^{h} \frac{S}{2h} e^{-i\omega t}dt = \frac{S}{2h(-i\omega)}\left(e^{-i\omega h} - e^{i\omega h}\right) = S\text{ sinc }\omega h.$$

From the graph of sinc in Figure 1, we see that for fixed h, the larger frequencies ω are weakly represented in the signal f_h. However, since $\lim_{x\to 0}$ sinc $x = 1$, it also follows

that for any given frequency ω, as $h \to 0$ the component of the signal with frequency ω becomes stronger. In fact, the smaller the duration h, the larger the range of the frequencies with nearly the same strength S. In acoustics, a pulse of high strength and small duration can be realized, for example, when two hard spheres collide at a high speed. In this case, all the frequencies in the audible range would be represented with nearly uniform strength and a sharp click is heard.

Perhaps the most celebrated result in communication theory is Shannon's sampling theorem. A function is said to be band limited with band width Ω if its Fourier transform vanishes $(F(\omega) = 0)$ outside the interval $(-\Omega, \Omega)$. Band limited functions are common as many devices effectively transmit only a certain range of frequencies. Shannon's theorem states that for a band limited function f,

$$ f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2\Omega}\right) \text{sinc } \pi \, (2\Omega x - n). $$

Proofs can be found in Hamming [9, page 557] and Rosie [13, page 53]. This remarkable result states that by sampling a band limited function only at the discrete set of points $\left\{ 0, \ \pm\dfrac{1}{2\Omega}, \ \pm\dfrac{2}{2\Omega}, \ldots \right\}$, we can reconstruct the entire function. Further, as Rosie states [13, p. 56], the function is "reconstituted by erecting at each sampling point a $\sin x / x$ function of magnitude equal to the sampled value at that point."

Further reading about the roles of sinc in the field of signal analysis appear in [2], [11], [12], and [13].

Approximation theory and numerical analysis. Long before the sampling theorem of Shannon appeared, E. T. Whittaker [15] noticed many important properties of the sinc function, and saw its role in the approximation of functions. Suppose the values of a function f on the real line are given at a discrete set of points, nh, $n = 0, \pm 1, \pm 2, \ldots$, for $h > 0$. Then f can be interpolated at these points by the function

$$ C(x) = \sum_{n=-\infty}^{\infty} f(nh)\text{sinc } \pi \left(\frac{x}{h} - n\right) $$

wherever the series converges. Since the roots of sinc are the nonzero multiplies of π, and sinc $0 = 1$, the function C takes on the same values as f at the points nh, $n = 0$, $n = \pm 1, \ldots$. The function C is called the Whittaker cardinal function for f, or the "sinc function" expansion of f. From Shannon's sampling theorem, we see that Whittaker's cardinal function will reproduce a band limited function when the spacing between data points is $h = 1/(2\Omega)$. The function C has been used in applications to the transmission of information [14, p. 166]. Moreover, Stenger [14], in an excellent article, analyzes many significant applications of the Whittaker cardinal function throughout numerical analysis, including the areas of interpolation theory, numerical integration, and solution of differential and integral equations. As Stenger notes, E. T. Whittaker used C to obtain alternate expressions of entire functions, and called it "a function of royal blood in the family of entire functions, whose distinguished properties separate it from its bourgeois brethren."

Conclusion. We have spent some time studying $\sin x / x$ and its several roles in mathematics and applications. The results of the previous section hint that, in some sense, it is a basic building block for a large class of functions. There are other

occurrences of sinc that we could have discussed. In mechanics and quantum physics, it arises in the solution of the wave equation [**8**, section 16.3]. In probability theory, sinc is the characteristic function of the uniform random variable on $(-1, 1)$.

We hope this article will stimulate readers to think about the prevalence of the sinc function. When we introduce $\sin x/x$ in the calculus course, preparing to find the derivatives of $\sin x$ and $\cos x$, we might point out that this function is significant in its own right, and that its use in calculus is not just an isolated occurrence. Many of our calculus students are engineering and science majors, and are likely to see this function again.

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